① Calculate $\int_{2}^{3} x^2 e^{x^3} dx =$

Use the substitution

$x^3 = y(x)$

$3x^2 = \frac{dy}{dx} \Rightarrow 3x^2 dx = dy$

$x^3 dx = \frac{dy}{3}$

x	2	3
y	8	27

$$= \int_{8}^{27} e^y \frac{dy}{3} = \frac{1}{3} \int_{8}^{27} e^y dy = \frac{1}{3} e^y \Big|_{y=8}^{y=27} =$$

$$= \boxed{\frac{1}{3}\left(e^{27} - e^{8}\right)}$$

---

② Calculate $\int_{0}^{1} \arcsin z \, dz =$

$\int_{a}^{b} u\,dv = uv\Big|_{a}^{b} - \int_{a}^{b} v\,du$

$dz = dv$

$z = v$

$u = \arcsin z$

$du = \frac{dz}{\sqrt{1-z^2}}$

$$= z\arcsin z\Big|_{0}^{1} - \underbrace{\int_{0}^{1} \frac{z\,dz}{\sqrt{1-z^2}}}_{I} = z\arcsin z\Big|_{0}^{1} - I$$

To integrate I, we let

$$1 - z^2 = w(z)$$
$$-2z\,dz = dw$$
$$z\,dz = -\frac{dw}{2}$$

z	0	1
w	1	0

Thus, $I = \int_0^1 \frac{z\,dz}{\sqrt{1-z^2}} = \int_1^0 -\frac{dw}{2} \cdot \frac{1}{\sqrt{w}} =$

$$= -\frac{1}{2}\int_1^0 \frac{dw}{\sqrt{w}} = \frac{1}{2}\int_0^1 \frac{dw}{\sqrt{w}} = \frac{1}{2}\int_0^1 w^{-\frac{1}{2}}\,dw =$$

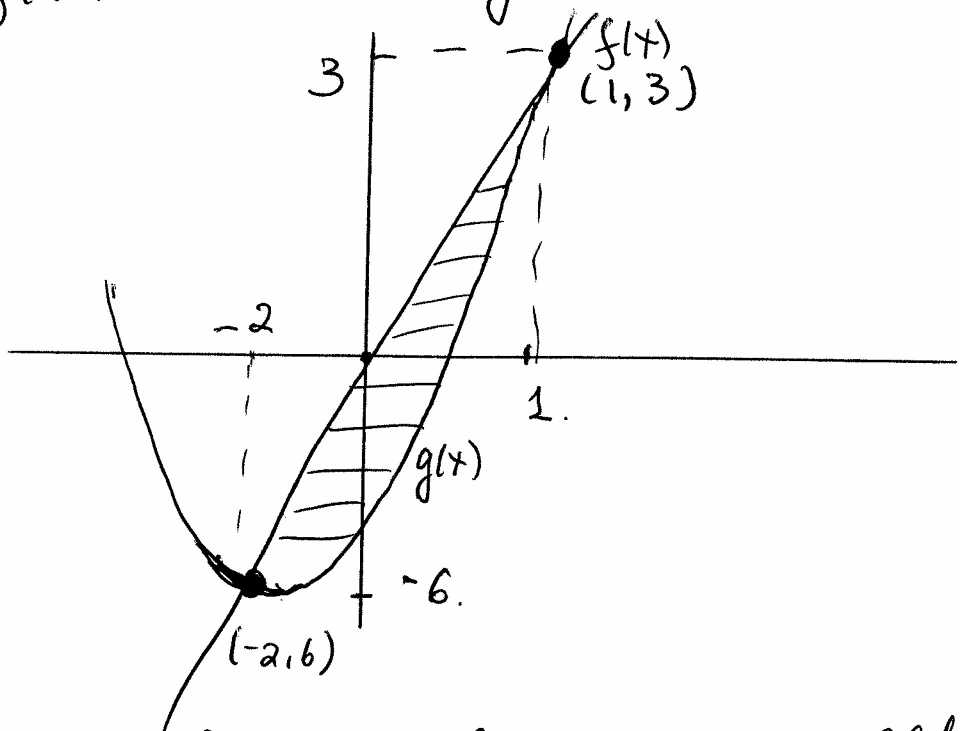
$$= \frac{1}{2}\cdot \frac{w^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Bigg|_0^1 = \frac{w^{\frac{1}{2}}}{2\cdot\frac{1}{2}}\Bigg|_0^1 = w^{\frac{1}{2}}\Big|_0^1 = 1.$$

Going back to the original integral, we have

$$\int_0^1 \arcsin z\,dz = z\arcsin z\Big|_0^1 - 1. = 1\cdot\arcsin 1 -$$

$$-0\cdot\arcsin 0 - 1 = 1\cdot\frac{\pi}{2} - 0 - 1 = \boxed{\frac{\pi}{2} - 1.}$$

③ Find the area of the region between $f(x) = 3x$ and $g(x) = (x+2)^2 - 6$



Solving $(x+2)^2 - 6 = 3x$ for x, we get the two points of intersection

$x^2 + 4x - 2 = 3x$

$x^2 + x - 2 = 0$

$D = 1 - 4(-2)1 = 9$

$x_1 = \frac{-1+3}{2} = 1$; $x_2 = \frac{-1-3}{2} = -2$

$g(x_1) = g(1) = 3 = f(1)$

$g(x_2) = g(-2) = -6 = f(-2)$.

The area of the region is given by

$$\int_{-2}^{1} |f(x) - g(x)|\,dx = \int_{-2}^{1} (f(x) - g(x))\,dx =$$

Since $f(x) - g(x) > 0$ for all $x \in (-2, 1)$

For example, take $\bar{x} = 0 \in (-2, 1)$ $f(0) = 0$

$g(0) = -2$

$f(0) > g(0)$

$$= \int_{-2}^{1} \left[3x - \left((x+2)^2 - 6 \right) \right] dx = \int_{-2}^{1} (-x^2 - x + 2) dx =$$

$$= -\frac{x^3}{3} \Big|_{-2}^{1} - \frac{x^2}{2} \Big|_{-2}^{1} + 2x \Big|_{-2}^{1} =$$

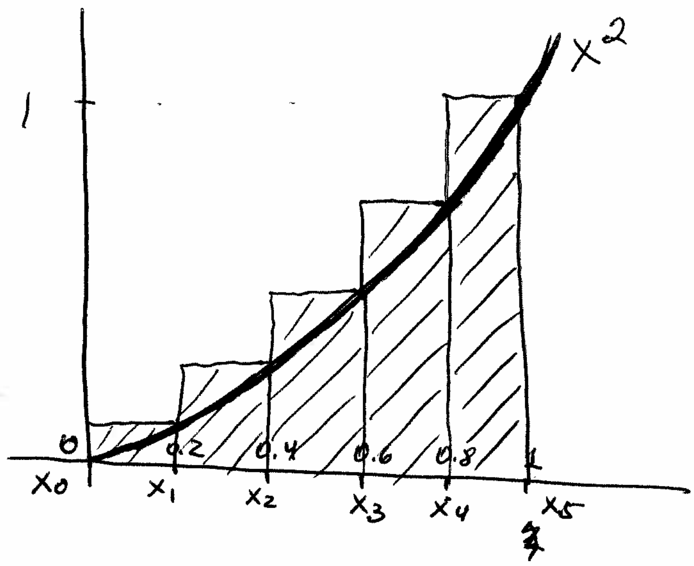
$$= -\frac{1}{3}\left(1^3 - (-2)^3 \right) - \frac{1}{2}\left(1^2 - (-2)^2 \right) + 2\left(1 - (-2) \right) =$$

$$= -\frac{1}{3}(9) - \frac{1}{2}(-3) + 6 = -3 + \frac{3}{2} + 6 = \frac{9}{2}$$

---

(4) Consider $f(x) = x^2$ on $[0,1]$. Partition $[0,1]$ into five equal subintervals and evaluate

(a) the RH Riemann sum, I_R.



$$\Delta x = \frac{b-a}{n} = \frac{1-0}{5} = 0.$$

$x_0 = a = 0$

$x_1 = x_0 + \Delta x = 0.2$

$x_2 = 0.4$

$x_3 = 0.6$

$x_4 = 0.8$

$x_5 = x_0 + 5 \cdot \Delta x = 1.$

$$I_R = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + f(x_4) \cdot \Delta x + f(x_5) \cdot \Delta x = \sum_{i=1}^{5} f(x_i) \cdot \Delta x =$$

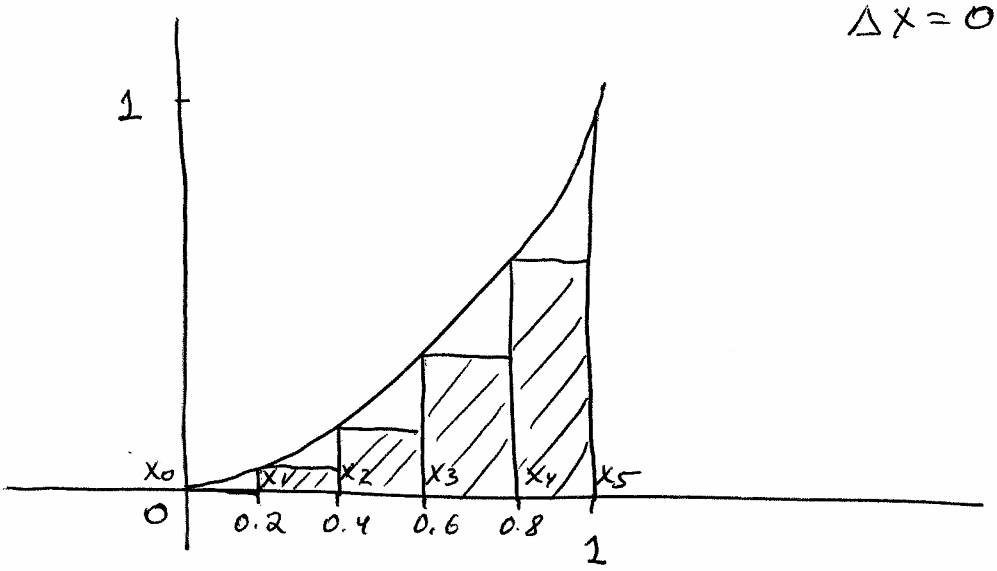
$$= f(0.2) \cdot 0.2 + f(0.4) \cdot 0.2 + f(0.6) \cdot 0.2 + f(0.8) \cdot 0.2 + f(1) \cdot 0.2 = (0.2)^2 \cdot 0.2 + (0.4)^2 \cdot 0.2 +$$

$$+ (0.6)^2 \cdot 0.2 + (0.8)^2 \cdot 0.2 + 1^2 \cdot 0.2 =$$

$= (0.04 + 0.16 + 0.36 + 0.64 + 1) \cdot 0.2 = 0.44$

(b) the LHS Riemann sum, I_L.

$\Delta x = 0.2.$



$$I_L = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x +$$
$$+ f(x_4) \cdot \Delta x = \sum_{i=0}^{4} f(x_i) \cdot \Delta x =$$
$$= 0 \cdot 0.2 + (0.2)^2 \cdot 0.2 + (0.4)^2 \cdot 0.2 + (0.6)^2 \cdot 0.2 +$$
$$+ (0.8)^2 \cdot 0.2 = (0 + 0.04 + 0.16 + 0.36 + 0.64) \cdot 0.2$$
$$= 0.24$$

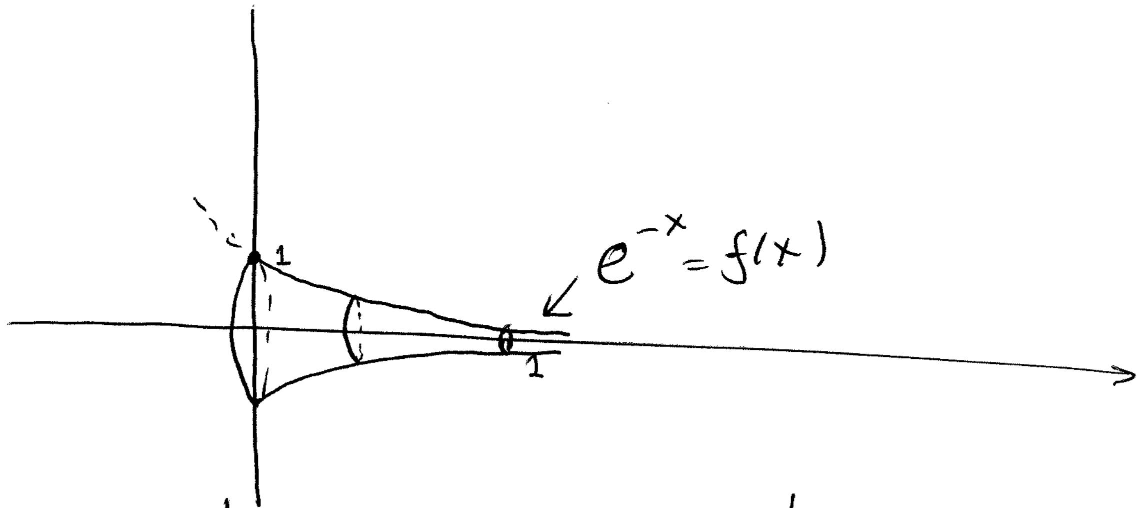
(c) $\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} \approx 0.33$

(d) $0.24 = I_L < \int_0^1 x^2 dx \approx 0.33 < I_R = 0.44$

I_R overestimates the actual value of the integral
I_L underestimates it.
It happens because $f(x) = x^2$ is an increasing function on $[0,1]$.

(5) $f(x) = e^{-x}$, find the volume of the solid of revolution

[$x = 0$, $x = 1$, around the x-axis].



$$\text{Volume} = \int_0^1 \pi [f(x)]^2 \, dx = \pi \int_0^1 \left(e^{-x}\right)^2 dx =$$

$$= \pi \int_0^1 e^{-2x} \, dx = \pi \left. \frac{e^{-2x}}{-2} \right|_{x=0}^{x=1} =$$

$$= -\frac{\pi}{2} \left[e^{-2} - e^{0} \right] = -\frac{\pi}{2} \left[\frac{1}{e^2} - 1 \right] =$$

$$= \frac{\pi}{2} \left[1 - \frac{1}{e^2} \right] \approx 1.36 \text{ (unit}^3\text{)}.$$