Student number:____________________, Total marks: __________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [5 points] Calculate the following integrals

$$\text{a)} \int_{-1}^{1} \frac{3y-4}{y^2-4y+4}\,dy \qquad \text{b)} \int_{1}^{e} \frac{1}{x[1+(\ln x)^2]}\,dx$$

3 a) $$\int_{-1}^{1} \frac{3y-4}{y^2-4y+4}\,dy = \int_{-1}^{1} \frac{3}{(y-2)}\,dy + \int_{-1}^{1} \frac{2}{(y-2)^2}\,dy = 3\ln|y-2|\Big|_{-1}^{1} - 2\,\frac{1}{y-2}\Big|_{-1}^{1}$$

$$= 3\ln|-1| - 3\ln|-3| - 2\left(\frac{1}{-1} - \frac{1}{-3}\right) = \frac{4}{3} - 3\ln 3$$

2 b) $$\int_{1}^{e} \frac{1}{x(1+(\ln x)^2)}\,dx = \int_{0}^{1} \frac{1}{1+y^2}\,dy = \arctan(1)$$

$y = \ln x$

Version 2:

a) $$\int_{-1}^{1} \frac{2y-5}{y^2-6y+9}\,dy = -2\ln(2) + \frac{1}{4}$$

b) $$= -\int_{1}^{1/e} \frac{1}{1+y^2}\,dy = \arctan(1) - \arctan(\tfrac{1}{e})$$

Version 3:

a) $$\int_{-1}^{1} \frac{y-2}{y^2-8y+16}\,dy = \ln\left|\frac{3}{5}\right| + \frac{4}{15}$$

b) $$\int_{0}^{1} \frac{e^x}{1+e^{2x}}\,dx = \arctan(e) - \arctan(1)$$

Question 2. [3 points] Solve the separable differential equation

$$\frac{dy}{dt} = \frac{7te^{-5t}}{y}$$

with initial condition $y(0) = 4$.

Separable: $y\,dy = 7te^{-5t}\,dt$

integrate: $\frac{y^2}{2} = 7\left[\frac{te^{-5t}}{-5} - \int \frac{e^{-5t}}{-5}\,dt\right]$

$= 7\left[-\frac{te^{-5t}}{5} - \frac{e^{-5t}}{25}\right] + C$

initial condition: $\frac{16}{2} = 7\left[-\frac{1}{25}\right] + C$

$8 = -\frac{7}{25} + C \qquad C = 8 + \frac{7}{25} = \frac{207}{25}$

Solution: $y(t) = \sqrt{-14\left(\frac{te^{-5t}}{5} + \frac{e^{-5t}}{25}\right) + \frac{414}{25}}$

Version 2: $y(t) = \sqrt{-6\left(\frac{te^{-2t}}{2} + \frac{e^{-2t}}{4}\right) + \frac{70}{4}}$

Version 3: $y(t) = \sqrt{-8\left(\frac{te^{-7t}}{7} + \frac{e^{-7t}}{49}\right) + \frac{792}{49}}$

Question 3. [4 points] Find the indefinite integral

$$\int \frac{5x+17}{x^2+5x-14}\,dx.$$

$x^2+5x-14 = (x-2)(x+7)$

$$\frac{5x+17}{x^2+5x-14} = \frac{3}{x-2} + \frac{2}{x+7}$$

$$\int \frac{5x+17}{x^2+5x-14}\,dx = 3\ln|x-2| + 2\ln|x+7| + C$$

Version 2:

$$\int \frac{5x-25}{x^2-9x+14}\,dx = \int\left(\frac{3}{x-2} + \frac{2}{x-7}\right)dx = 3\ln|x-2| + 2\ln|x-7| + C$$

Version 3:

$$\int \frac{5x+4}{x^2+3x-10}\,dx = \int \frac{2}{x-2} + \frac{3}{x+5}\,dx = 2\ln|x-2| + 3\ln|x+5| + C$$

Question 4. [6 points] For each of the following improper integrals, determine whether it converges, and determine its value if it does.

a) $\displaystyle\int_1^{\infty} \frac{5+2x}{x^2}dx$ b) $\displaystyle\int_1^{2} \frac{4}{\sqrt[3]{x-1}}dx$

3 a) $$\int_1^{\infty} \frac{5}{x^2}\,dx + \int_1^{\infty} \frac{2}{x}\,dx > \int_1^{\infty} \frac{2}{x}\,dx$$

now $$\int_1^{\infty} \frac{2}{x}\,dx = \lim_{T\to\infty} 2\int_1^{T} \frac{1}{x}\,dx = \lim_{T\to\infty} 2\ln|T| = \infty$$

This integral diverges. [Same for all three versions]

3 b) $$\int_1^{2} \frac{4}{(x-1)^{1/3}}\,dx = \lim_{\varepsilon\to 1} \int_{\varepsilon}^{2} \frac{4}{(x-1)^{1/3}}\,dx$$

$$= \lim_{\varepsilon\to 1^+} 4\cdot\frac{3}{2}(x-1)^{2/3}\Big|_{\varepsilon}^{2} = \lim_{\varepsilon\to 1^+} \left[6 - 6(\varepsilon-1)^{2/3}\right] = 6 \text{ converges}$$

Version 2: $$\int_2^{3} \frac{3}{(x-2)^{1/3}}\,dx = \lim_{\varepsilon\to 2^+} 3\cdot\frac{3}{2}(x-2)^{1/3}\Big|_{\varepsilon}^{3} = \frac{9}{2} \text{ converges}$$

Version 3: $$\int_1^{2} \frac{7}{(x-1)^{1/4}}\,dx = \lim_{\varepsilon\to 1^+} 7\cdot\frac{4}{3}(x-1)^{3/4}\Big|_{\varepsilon}^{2} = \frac{28}{3} \text{ converges}$$

Question 5. [4 points] Consider the functions $f(x) = 5x^2$ and $g(x) = -9x + 2$.
a) Find the points of intersection of the graphs of f and g.
b) Find the area enclosed between the two graphs between these two points of intersection.

a) $f(x) = g(x) \longleftrightarrow 5x^2 + 9x - 2 = 0$

$\longleftrightarrow (x+2)(5x-1) = 0$

Points of intersection: $x = -2$ and $x = \frac{1}{5}$

b) In the interval $(-2, \frac{1}{5})$ we have $g(x) > f(x)$

Area between f and g:

$$\int_{-2}^{1/5} |f(x) - g(x)|\, dx = \int_{-2}^{1/5} g(x) - f(x)\, dx$$

$$= \int_{-2}^{1/5} -5x^2 - 9x + 2\, dx = -\frac{5}{3}x^3 - \frac{9}{2}x^2 + 2x \Big|_{-2}^{1/5}$$

$$= -\frac{5}{3}\frac{1}{125} - \frac{9}{2}\frac{1}{25} + \frac{2}{5} - \left(\frac{40}{3} - 18 - 4\right)$$

$$= -\frac{1}{75} - \frac{9}{50} + \frac{2}{5} - \frac{40}{3} + 18 + 4 = 22 + \frac{-2 - 27 + 60 - 2000}{150}$$

$$= 22 - \frac{1969}{150} \approx 8.87$$

Question 5,

Version 2:

$$5x^2 - 9x - 2 = (5x+1)(x-2) = 0$$

intersection: $x = -\frac{1}{5}$, $x = 2$

and in the interval $g(x) > f(x)$

Area: $$\int_{-1/5}^{2} -5x^2 + 9x + 2\, dx = -\frac{5}{3}x^3 + \frac{9}{2}x^2 + 2x \Big|_{-1/5}^{2}$$

$$= 22 - \frac{1969}{150} \approx 8.87$$

Version 3:

$$5x^2 + 14x - 3 = (5x-1)(x+3) = 0$$

intersection: $x = \frac{1}{5}$, $x = -3$

and in the interval $g > f$

Area: $$\int_{-3}^{1/5} -5x^2 - 14x + 3\, dx = -\frac{5}{3}x^3 - 7x^2 + 3x \Big|_{-3}^{5}$$

$$= 27 + \frac{23}{75} \approx 27.307$$

Question 6. [4 points] Zombies have invaded campus! They recruit more of the undead to their ghoulish ranks at rate

$$\frac{dz}{dt} = f(z) = -8z^3 + 64800z,$$

where t is time and z is the number of zombies.

a) Determine all biologically meaningful equilibria of this dynamical system.
b) Determine the stability of each equilibrium in (a), using the derivative test.
c) Draw a phase-line diagram for $z < 150$.

a) steady states: $f(z) = 0$ $\qquad -8z^3 + 64800z = 0$

$$8z(-z^2 + 8100) = 0$$

$$z = 0, \quad z = \pm 90$$

biologically meaningful: $z_1 = 0$, $z_2 = 90$

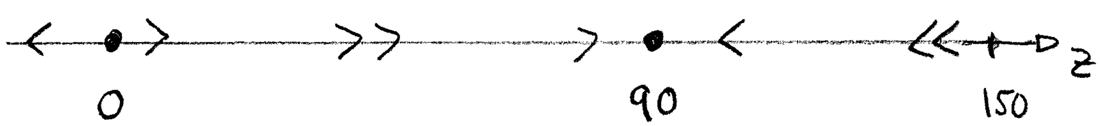
b) $f'(z) = -24z^2 + 64800$

$f'(0) = +64800 > 0$ $\qquad \Rightarrow z_1 = 0$ is unstable

$f'(90) = -24 \cdot 8100 + 8 \cdot 8100 = -16 \cdot 8100 < 0$

$\Rightarrow z_2 = 90$ is stable

c)



Version 2: same z_2 , Version 3: $z_2 = 110$

Question 7. [4 points] Determine the average value of $f(x) = x\sin(2x)$ over the range $0 \le x \le 2\pi$.

Average : $$\frac{1}{2\pi} \int_0^{2\pi} x \sin(2x)\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{x\cos(2x)}{2}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{\cos(2x)}{2}\,dx \right]$$

$$= \frac{1}{2\pi}\left[\frac{-x\cos(2x)}{2}\Big|_0^{2\pi} + \frac{\sin(2x)}{4}\Big|_0^{2\pi} \right]$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi \cdot 1}{2} - 0 + 0 - 0 \right]$$

$$= -\frac{1}{2}$$

Version 2 : $$\frac{1}{2\pi}\int_0^{2\pi} x \sin(3x)\,dx = -\frac{1}{3}$$

Version 3: $$\frac{1}{2\pi}\int_0^{2\pi} x \sin(4x)\,dx = -\frac{1}{4}$$

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