University of Ottawa
MAT 1330 D Midterm Exam
February 24, 2014. Duration: 80 minutes.
Instructor: Yuxiang Zhang

Surname: ____

Given Names: ____

PLEASE NOTE:

Do not write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: **you do not** have to proceed through the paper in the order given.

You have 80 minutes to complete this exam.

This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device is not permitted.

Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed.

The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
Where it is possible to check your work, do so.

Please do not detach the pages. There are 5 questions (8 pages) in this exam.

Good luck!

Student number:______________, Total marks:______________out of 40

Problems	1	2	3	4	5
Marks					

Question 1. [3 points] Find the value of a so that the following function is continuous

$$f(x) = \begin{cases} \dfrac{x^2 - 4x + 3}{x - 1} & \text{if } x < 1, \\ a + \sin(\pi x) & \text{if } x \geq 1. \end{cases}$$

It is obvious that $f(x)$ is continuous if $f(x)$ is continuous at $x=1$.

We need $\lim_{x \to 1} f(x) = f(1)$ to make sure $f(x)$ is continuous.

$$\underline{\underline{\lim_{x \to 1^-} f(x)}} = \lim_{x \to 1^-} \frac{x^2-4x+3}{x-1} = \lim_{x \to 1^-} \frac{(x-3)(x+1)}{x-1}$$

$$= \lim_{x \to 1^-} (x-3) = -2$$

$$\underline{\underline{\lim_{x \to 1^+} f(x)}} = \lim_{x \to 1^+} (a + \sin \pi x) = a = f(1).$$

$\Rightarrow$ $\underline{a = -2}$ such that $f(x)$ is continuous.

Question 2. [7 points] Consider the following DTDS for the concentration of the drug in the body of a patient.

$$x_{t+1} = 0.5x_t + 2$$

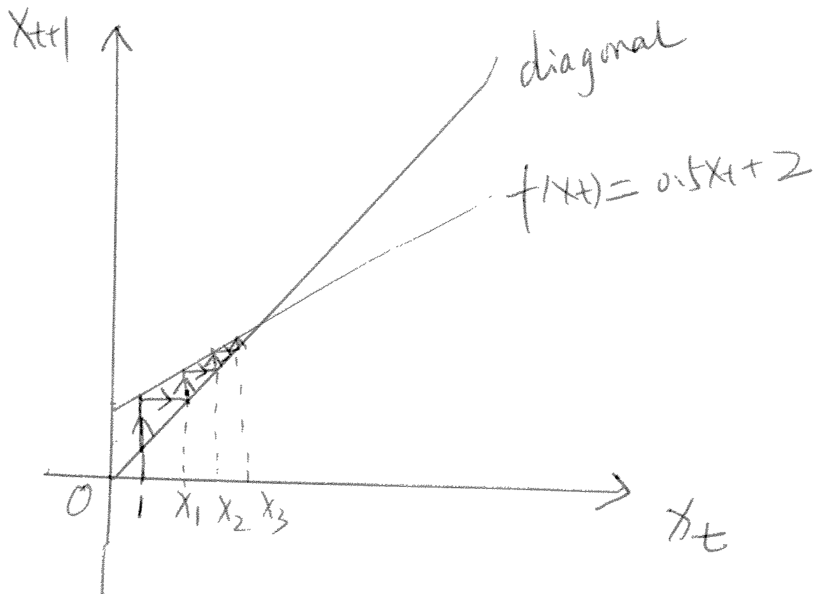
(a) [1 point] The updating function of the DTDS is $f(x) =$ 0.5x + 2

(b) [1 point] The equilibrium point of the DTDS is $x^* =$ 4

(c) [2 points] Give the solution formula for the DTDS with the initial condition x_0 :

$x_t =$ $0.5^t x_0 + 4(1 - 0.5^t)$

(d) [2 points] Graph the updating function and cobweb diagram of the DTDS with $x_0 = 1$ for at least three steps.



(e) [1 point] Is the equilibrium point is stable or unstable? Stable

Questions 3. [15 points] Find the derivative of the following functions with respect to variable x.

(a) $f(x) = \frac{1+x}{1-x} + \sqrt{x-1}$

$$f'(x) = \left(\frac{1+x}{1-x}\right)' + \left(\sqrt{x-1}\right)'$$

$$= \frac{(1+x)'(1-x) - (1+x)(1-x)'}{(1-x)^2} + \frac{1}{2}(x-1)^{-\frac{1}{2}}(x-1)'$$

$$= \frac{1-x+1+x}{(1-x)^2} + \frac{1}{2\sqrt{x-1}}$$

$$= \frac{2}{(1-x)^2} + \frac{1}{2\sqrt{x-1}}$$

(b) $f(x) = x\sin(x+1) + e^{x^2}$

$$f'(x) = x'\sin(x+1) + x\left(\sin(x+1)\right)' + e^{x^2} \cdot (x^2)'$$

$$= \sin(x+1) + x\cos(x+1) + 2x\,e^{x^2}$$

(c) $f(x) = \ln(\frac{1}{x}) + \tan(x^2)$

$$f'(x) = (\ln\tfrac{1}{x})' + (\tan x^2)'$$

$$= \frac{1}{\frac{1}{x}} \cdot (\tfrac{1}{x})' + \sec^2(x^2) \cdot (x^2)'$$

$$= x \cdot (x^{-1})' + \sec^2(x^2) \cdot 2x$$

$$= x(-1x^{-2}) + 2x \cdot \sec^2(x^2)$$

$$= \frac{-1}{x} + 2x \cdot \sec^2(x^2)$$

Question 4. [10 points] Consider the function $f(x) = \frac{x}{x-1}$

(a) [1 point] Find the domain of f.

$$D_f = (-\infty, 1) \cup (1, +\infty)$$

(b) [1.5 points] Find the limits of f as x approaches $\pm\infty$. Try to write down the horizontal asymptotes of $f(x)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x-1} = \lim_{x\to\infty} \frac{x/x}{x/x - 1/x} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x-1} = \lim_{x\to-\infty} \frac{x/x}{x/x - 1/x} = 1$$

$\Rightarrow$ $y=1$ is the asymptote of function $f(x)$.

(c) [2 points] Are there points where $f(x)$ is not continuous? If yes, try to find the one-side limits as x approaches those points from left and right sides. Try to find the vertical asymptotes of $f(x)$.

Since $x=1$ is not in the Domain of $f(x)$,

$f(x)$ is not continuous at $x=1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x}{x-1} = -\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x}{x-1} = +\infty$$

$\Rightarrow$ $x=1$ is the vertical asymptote of $f(x)$.

(d) [2 points] Find the increasing and decreasing intervals of $f(x)$. Are there critical points?

Since $f'(x) = \left(\frac{x}{x-1}\right)'$

$$= \frac{x'(x-1) - x(x-1)'}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

We know $f'(x) < 0$ for all $x \in D_f$, that

is $f(x)$ is always decreasing in the domain.

$\Rightarrow$ No critical point.

(e) [2 points] Find the intervals where $f(x)$ is concave up or concave down . Are there any points of inflection?

Since $f''(x) = (f'(x))' = \left(\frac{-1}{(x-1)^2}\right)'$

$$= \frac{(-1)'(x-1)^2 - (-1)((x-1)^2)'}{(x-1)^4}$$

$$= \frac{2(x-1)}{(x-1)^4} = \frac{2}{(x-1)^3}$$

$\Rightarrow$ $f''(x) > 0$, $x \in (1, +\infty)$, and

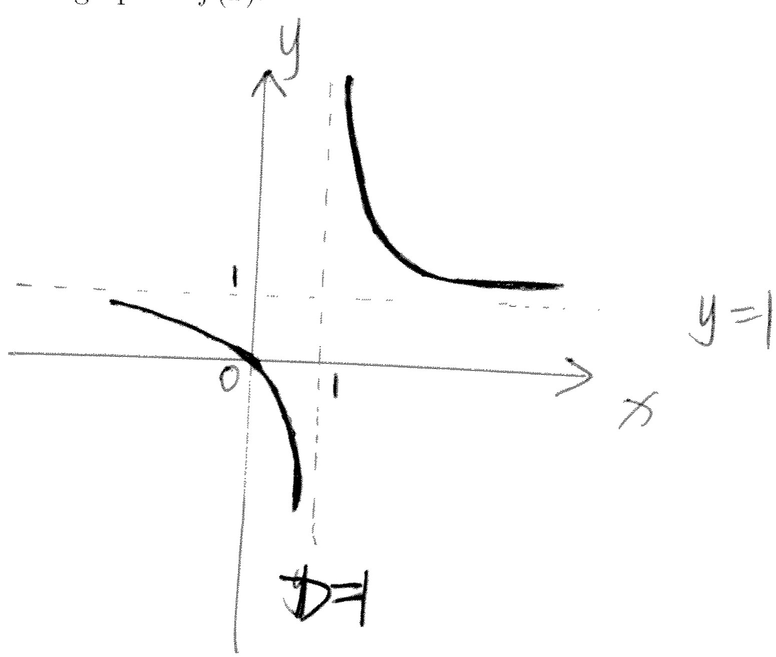
$f''(x) < 0$, $x \in (-\infty, 1)$.

$\Rightarrow$ $f(x)$ is concave up in $(1, +\infty)$,

$f(x)$ is concave down in $(-\infty, 1)$.

However, $f(x)$ has no inflection point since $f(x)$ is not defined at $x=1$.

(f) [1.5] Sketch the graph of $f(x)$.



Question 5. [5 points] Consider the function $f(x) = e^{2x}$.
(a) Find the tangent line approximation with base point $a = 0$.

(b) Find a Taylor polynomial of degree three (i.e. cubic polynomial) with the base point $a = 0$.

Since $f(0) = e^0 = 1$

$$f'(0) = (e^{2x})'\big|_{x=0} = 2e^{2x}\big|_{x=0} = 2$$

$$f''(0) = (2e^{2x})'\big|_{x=0} = 4e^{2x}\big|_{x=0} = 4$$

$$f'''(0) = (4e^{2x})'\big|_{x=0} = 8e^{2x}\big|_{x=0} = 8,$$

(a) we have tangent line approximation is given

$$y = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$$\Rightarrow y = 1 + 2x$$

(b) Talyor polynomial of degree three is given by

$$y = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\Rightarrow y = 1 + 2x + 2x^2 + \frac{8}{6}x^3$$

$$\Rightarrow y = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$