Complex Eigenvalues and Eigenvectors

For a complex scalar λ,

$$\det(A - \lambda I) = 0 \Longleftrightarrow Ax = \lambda x$$

for some non-zero vector x in C^n.

λ is called a (complex) eigenvalue and
x is called a (complex) eigenvector.

If x is a complex vector in C^n, then the vector $\overline{x}$, whose entries are the complex conjugates of the entries in x, is called the complex conjugate of x. Thus,

if $x =$Re $x + i$ Im x, then $\overline{x} =$Re $x - i$ Im x.

Real matrices with complex eigenvalues:

Let A be an $n \times n$ matrix whose entries are real. If λ is an eigenvalue of A with a corresponding eigenvector x in C^n, then $\overline{\lambda}$ is also an eigenvalue of A with the corresponding eigenvector $\overline{x}$:

$$\begin{aligned} Ax &= \lambda x \\ \overline{Ax} &= \overline{\lambda x} \\ \overline{A}\overline{x} &= \overline{\lambda}\overline{x} \\ A\overline{x} &= \overline{\lambda}\overline{x} \end{aligned}$$

This shows that if A is a real matrix, its complex eigenvalues occur in conjugate pairs.

Example: Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A, and a basis for each eigenspace in C^2.

b) Diagonalize A.

Solution:

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}, \\ \det(A-\lambda I) &= (1-\lambda)(3-\lambda)+2 \\ &= \lambda^2 - 4\lambda + 5 = 0, \end{aligned}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{4 \pm \sqrt{(-4)^2 - 4\cdot 1 \cdot 5}}{2} \\ &= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i. \end{aligned}$$

$\lambda_1 = 2 + i$, $\lambda_2 = 2 - i$.

Eigenvectors for $\lambda_1 = 2 + i$:

$$(A - \lambda_1 I)X = 0 \Longleftrightarrow (A - (2+i)I)X = 0$$

$$\begin{bmatrix} 1-(2+i) & -2 \\ 1 & 3-(2+i) \end{bmatrix} = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1-i & -2 & 0 \\ 1 & 1-i & 0 \end{array}\right] R_1 \longleftrightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 1-i & 0 \\ -1-i & -2 & 0 \end{array}\right] R_2' = R_2 + (1+i)R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (-1+i)t \\ t \end{bmatrix} = t\begin{bmatrix} -1+i \\ 1 \end{bmatrix}.$$

For the eigenvalue $\lambda_1 = 2 + i$,

$v_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ is an eigenvector, and

$E_1 =$Span $\left\{\begin{bmatrix} -1+i \\ 1 \end{bmatrix}\right\}$ is the complex eigenspace.

Eigenvectors for $\lambda_2 = 2 - i$:

$$\begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since $\lambda_2 = \overline{\lambda_1}$, an eigenvector corresponding to eigenvalue λ_2 is

$$v_2 = \overline{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - i \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Complex eigenspace corresponding to

$\lambda_2 = 2 - i$ is $E_2 =$Span $\left\{ \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \right\}$.

A basis for E_2 is $\left\{ \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \right\}$.

b) $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}.$$

Check: $A = PDP^{-1} \Longleftrightarrow AP = PD$

$$AP = PD = \begin{bmatrix} -3+i & -3-i \\ 2+i & 2-i \end{bmatrix}$$

If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a, b are real and $b \neq 0$, then the eigenvalues of A are $\lambda = a \pm bi$:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 \\ &= \lambda^2 - 2a\lambda + a^2 + b^2. \end{aligned}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{2a \pm \sqrt{(-2a)^2 - 4(a^2+b^2)}}{2} \\ &= \frac{2a \pm \sqrt{-4b^2}}{2} = \frac{2a \pm 2bi}{2} = a \pm bi. \end{aligned}$$

<u>Eigenvectors for $\lambda_1 = a - ib$:</u>

$$\begin{aligned} A - (a-ib)I &= \begin{bmatrix} a-(a-ib) & -b \\ b & a-(a-ib) \end{bmatrix} \\ &= \begin{bmatrix} ib & -b \\ b & ib \end{bmatrix} \sim \begin{bmatrix} ib & -b \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Corresponding eigenvectors are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2/i \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad x_2 \neq 0.$$

An eigenvector corresponding to eigenvalue $\lambda_1 = a - ib$ is $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Since $\lambda_2 = \overline{\lambda_1}$, an eigenvector corresponding to eigenvalue $\lambda_2 = a + ib$ is $v_2 = \overline{v_1} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Thus, the eigenvalues of $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, ($a$, b are real and $b \neq 0$) are $\lambda_1 = a - ib$ and $\lambda_2 = a + ib$ with the corresponding eigenvectors

$$v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix},$$

respectively.

Let $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} a - ib & 0 \\ 0 & a + ib \end{bmatrix}$.

$$\begin{aligned} P^{-1} &= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1} = \frac{-1}{2i} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} \\ &= \frac{i}{2} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ &= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a - ib & 0 \\ 0 & a + ib \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix} \end{aligned}$$

Also, if $r = |\lambda| = \sqrt{a^2 + b^2}$, then

$$\begin{aligned} A &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} \\ &= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \end{aligned}$$

where θ is the angle between the positive x-axis and the ray from $(0,0)$ through (a, b).

The transformation $x \longrightarrow Ax$ may be viewed as the composition of a rotation through the angle θ and a scaling by $|\lambda|$.

Example: Let $A = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$.

Then, $a = 3$, $b = 2$ and $\lambda_{1,2} = 3 \pm 2i$.

For $\lambda_1 = 3-2i$, $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

For $\lambda_2 = 3+2i$, $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\begin{aligned} A &= \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3-2i & 0 \\ 0 & 3+2i \end{bmatrix} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1}. \end{aligned}$$

Also,

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$$

Theorem: Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and associated eigenvector v in C^2. Then,

$$A = PCP^{-1}, \text{ where}$$

$$P = [\text{Re}v \quad \text{Im}v] \quad \text{and} \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Example: The matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ is a real 2×2 matrix with a complex eigenvalue $\lambda = 2 - i$ ($\mathrm{Im}\lambda \neq 0$) and associated eigenvector

$$v = \begin{bmatrix} -1 - i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ in } C^2,$$

$$A = PCP^{-1} \text{ where}$$

$$P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Remark: With the complex eigenvalue $\overline{\lambda} = 2 + i = 2 - (-1)i$ and associated eigenvector

$$\overline{v} = \begin{bmatrix} -1 + i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ in } C^2:$$

$$A = PCP^{-1} \text{ where}$$

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$. Then,

$$\det(A - \lambda I) \;=\; -\lambda(4-\lambda) + 8 = \lambda^2 - 4\lambda + 8.$$

$\lambda = 2 \pm 2i$.

<u>Eigenvectors for $\lambda_1 = 2 + 2i$:</u>

$$A - (2+2i)I \;=\; \begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix} \; R_2' = R_2 + (-2+2i)R_1$$

$$\sim \begin{bmatrix} -2-2i & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1+i & -1/2 \\ 0 & 0 \end{bmatrix}$$

Corresponding eigenvectors are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;=\; \begin{bmatrix} \frac{x_2}{2(1+i)} \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} (\frac{1-i}{4}) \\ 1 \end{bmatrix} = \frac{x_2}{4} \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$$

$$= \; t \begin{bmatrix} 1-i \\ 4 \end{bmatrix} = t \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right).$$

For $P = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$,

$A = PCP^{-1}$.

Example: Let $A = \begin{bmatrix} 0 & 3+4i \\ 3-4i & 0 \end{bmatrix}$.

a) Find the eigenvalues of A, and a basis for each eigenspace in C^2.

b) Diagonalize A.

Solution:

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} -\lambda & 3+4i \\ 3-4i & -\lambda \end{vmatrix} \\ &= \lambda^2 - (3+4i)(3-4i) \\ &= \lambda^2 - (9+16) = \lambda^2 - 25 = 0. \end{aligned}$$

Thus, $\lambda_1 = 5$ and $\lambda_2 = -5$.

<u>Eigenvectors for $\lambda_1 = 5$:</u>

$$(A-5I)X = 0 \Longleftrightarrow \left[\begin{array}{cc|c} -5 & 3+4i & 0 \\ 3-4i & -5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & \frac{-3-4i}{5} & 0 \\ 3-4i & -5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{-3-4i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3+4i}{5}t \\ t \end{bmatrix} = \frac{t}{5}\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}, \ t \in R.$$

For the eigenvalue $\lambda_1 = 5$,

$v_1 = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$ is an eigenvector, and

$E_1 = \text{Span}\left\{\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}\right\}$ is the eigenspace.

Eigenvectors for $\lambda_2 = -5$:

$$(A+5I)X = 0 \Longleftrightarrow \left[\begin{array}{cc|c} 5 & 3+4i & 0 \\ 3-4i & 5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & \frac{3+4i}{5} & 0 \\ 3-4i & 5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3+4i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3+4i}{5}t \\ t \end{bmatrix} = \frac{t}{5}\begin{bmatrix} -(3+4i) \\ 5 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = -5$,

$v_2 = \begin{bmatrix} -3-4i \\ 5 \end{bmatrix}$ is an eigenvector, and

$E_2 = \text{Span}\left\{\begin{bmatrix} -3-4i \\ 5 \end{bmatrix}\right\}$ is the eigenspace.

Note that the matrix A has complex entries,
eigenvalues of A are real,
corresponding eigenvectors are complex, and
$v_2 \neq \overline{v_1}$.

$A = PDP^{-1}$, where

$$P = \begin{bmatrix} 3+4i & -3-4i \\ 5 & 5 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$

Check: $A = PDP^{-1} \Longleftrightarrow AP = PD.$

$$AP = PD = \begin{bmatrix} 15+20i & 15+20i \\ 25 & -25 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} i & 0 & 1 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$.

a) Find the eigenvalues of A, and a basis for each eigenspace.

b) If possible, diagonalize A.

Solution: $\lambda = i$ is the only eigenvalue of A.

$$A - iI = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} (A - iI)X &= 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \\ X &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_1, x_2 \in R. \end{aligned}$$

The eigenspace for $\lambda = i$ is $E_i = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Since A is a 3×3 matrix and has only two linearly independent eigenvectors, A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$.

If possible, diagonalize A.

Solution:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & -4 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -2-\lambda \end{vmatrix} \\
&= (1-\lambda) \begin{vmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{vmatrix} \\
&= (1-\lambda)\Big((2-\lambda)(-2-\lambda)+8\Big) \\
&= (1-\lambda)(\lambda^2+4) \\
&= (1-\lambda)(\lambda+2i)(\lambda-2i) \\
\det(A-\lambda I) &= 0 \Longleftrightarrow (1-\lambda)(\lambda+2i)(\lambda-2i) = 0,
\end{aligned}$$

$\lambda_1 = 1$, $\lambda_2 = -2i$, $\lambda_3 = 2i$.

<u>Eigenvectors for $\lambda_1 = 1$:</u>

$$A - I = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A-I)X \;=\; 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_2 \in R.$$

Eigenvectors for $\lambda_2 = -2i$:

$$A + 2iI \;=\; \begin{bmatrix} 2+2i & 0 & -4 \\ 0 & 1+2i & 0 \\ 2 & 0 & -2+2i \end{bmatrix} \quad \begin{array}{l} R_1' = \frac{1}{2}\, R_1 \\ \\ R_3' = \frac{1}{2}\, R_3 \end{array}$$

$$\sim \begin{bmatrix} 1+i & 0 & -2 \\ 0 & 1+2i & 0 \\ 1 & 0 & -1+i \end{bmatrix} \quad R_1 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -1+i \\ 0 & 1+2i & 0 \\ 1+i & 0 & -2 \end{bmatrix} \quad R_3' = R_3 - (1+i)R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -1+i \\ 0 & 1+2i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 2iI)X = 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix}.$$

Since A is a real matrix and $\lambda_3 = \overline{\lambda_2}$, an eigenvector for $\lambda_3 = 2i$ is

$$Y = \overline{\begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1+i \\ 0 \\ 1 \end{bmatrix}.$$

$A = PDP^{-1}$, where

$$P = \begin{bmatrix} 0 & 1-i & 1+i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 2i \end{bmatrix}.$$

Homework: Let $A = \begin{bmatrix} -5 & 6 & 2 \\ -3 & 4 & 1 \\ -5 & 5 & 2 \end{bmatrix}$.
If possible, diagonalize A.

Ans: The eigenvalues are

$\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = -i$
with corresponding eigenvalues

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 2 \\ 1 \\ 2-i \end{bmatrix}, \; v_3 = \begin{bmatrix} 2 \\ 1 \\ 2+i \end{bmatrix}.$$