STAT 349 Sample Questions for Midterm Exam

1. A and B play games together. In each game $P\{\text{A wins}\} = p$. Whoever wins two more games than the other wins overall.
   (a) Find the probability that A wins overall.
   (b) Find the expected value of total number of games that A wins.

   Outline of solution:
   (a) Let $P_A = P\{\text{A wins overall}\}$. Condition on the first two outcomes,
$$P_A = p^2 + 2p(1-p)P_A.$$
Solve it for P_A.
   (b) Let μ_A be the expected value of total number of games that A wins. Then
$$\mu_A = 2p^2 + 2p(1-p)(1+\mu_A).$$

2. Three guests, A, B and C, have been invited to a party. The party begins at time 0. The arrival times of A, B and C are independent. The arrival times of A and B are exponential random variables with mean 1, and the arrival time of C is uniformly distributed between 0 and 1.

   (a) Find the probability that C arrives before A.
   (b) Find the expected number of guests who arrive after C.

   Solution:
   (a) Denote by X_A and X_C the arrival times of A and C, respectively. Let f_A be the p.d.f. of A. Then
$$\begin{aligned}
P(X_C < X_A) &= \int_{-\infty}^{\infty} P(X_C < X_A | X_A = t) f_A(t)dt \\
&= \int_0^1 t \cdot e^{-t}dt + \int_1^{\infty} 1 \cdot e^{-t}dt \\
&= -te^{-t}|_0^1 + \int_0^1 e^{-t}dt + \int_1^{\infty} e^{-t}dt \\
&= 1 - e^{-1} \\
&= 0.63212.
\end{aligned}$$

(b) Denote by N the number of guests arriving after C. Let f_C be the p.d.f. of C. Then

$$\begin{aligned}
E[N] &= \int_{-\infty}^{\infty} E[N|X_C = t) f_C(t) dt \\
&= \int_0^1 E[N|X_C = t) dt \\
&= \int_0^1 2e^{-t} dt, \text{ since } N|X_C = t \text{ is Binomial } (2, P(X_A > t)), \\
&= 2(1 - e^{-1}) \\
&= 1.26424.
\end{aligned}$$

Alternatively: $N = I(X_A > X_C) + I(X_B > X_C)$, so that $E(N) = P(X_A > X_C) + P(X_B > X_C) = 2P(X_A > X_C) = 2 \times 0.63212 = 1.26424$.

3. A balanced 6-sided die is continually rolled until the pattern 2, 5 appears for the first time. Find the expected number of rolls made.

Solution: Denote by N the number of rolls made until the pattern 2, 5 appears for the first time. Then

$$E[N] = E[N|2](1/6) + E[N| \neq 2](5/6) = E[N|2](1/6) + (1 + E[N])(5/6),$$

which implies that

$$E[N] = E[N|2] + 5. \tag{1}$$

Further,

$$\begin{aligned}
E[N|2] &= E[N|2,2](1/6) + E[N|2,5](1/6) + E[N|2, \neq 5](4/6) \\
&= (1 + E[N|2])(1/6) + (2/6) + (2 + E[N])(4/6),
\end{aligned}$$

which implies that

$$E[N|2] = (11 + 4E[N])/5. \tag{2}$$

By (**??**) and (**??**), we get

$$E[N] = 5 + (11 + 4E[N])/5.$$

Therefore $E[N] = 36$.

4. A Markov chain $\{X_n, n = 0, 1, 2, ...\}$ with states $0, 1, 2$ has a transition matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

If $P\{X_0 = 0\} = \frac{1}{2} = P\{X_0 = 1\}$,
(a) find $E[X_2^2]$;
(b) find $E[X_1 \,|\, X_2 = 2]$.

Outline of solution:
(a)

$$\begin{aligned} E[X_2^2] &= 1^2 \times P\{X_2 = 1\} + 2^2 \times P\{X_2 = 2\} \\ &= 1^2 \times (P\{X_0 = 0\}p_{01}^2 + P\{X_0 = 1\}p_{11}^2) \\ &\quad + 2^2 \times (P\{X_0 = 0\}p_{02}^2 + P\{X_0 = 1\}p_{12}^2). \end{aligned}$$

(b)

$$\begin{aligned} E[X_1 \,|\, X_2 = 2] &= 1 \times P\{X_1 = 1 \,|\, X_2 = 2\} + 2 \times P\{X_1 = 2 \,|\, X_2 = 2\} \\ &= 1 \times \frac{P\{X_1 = 1, X_2 = 2\}}{P\{X_2 = 2\}} + 2 \times \frac{P\{X_1 = 2, X_2 = 2\}}{P\{X_2 = 2\}} \\ &= 1 \times \frac{P\{X_0 = 0\}p_{01}p_{12} + P\{X_0 = 1\}p_{11}p_{12}}{P\{X_0 = 0\}p_{02}^2 + P\{X_0 = 1\}p_{12}^2} \\ &\quad + 2 \times \frac{P\{X_0 = 0\}p_{02}p_{22} + P\{X_0 = 1\}p_{12}p_{22}}{P\{X_0 = 0\}p_{02}^2 + P\{X_0 = 1\}p_{12}^2}. \end{aligned}$$

5. Consider the Ehrenfest urn model in which M molecules are distributed between two urns, and at each time-point one of the molecules is chosen at random and is then removed from its urn and placed in the other one. Let X_n denote the number of molecules in urn 1 after the n-th switch and let $\mu_n = E(X_n)$. Show that

$$\mu_{n+1} = 1 + \left(1 - \frac{2}{M}\right)\mu_n$$

and hence that

$$\mu_n = \frac{M}{2} + \left(\frac{M-2}{M}\right)^n \left(E(X_0) - \frac{M}{2}\right).$$

Solution:

We have for $i, j = 0, 1, \ldots, M$,

$$\begin{aligned}
P(X_{n+1} = i-1|X_n = i) &= i/M, \\
P(X_{n+1} = i+1|X_n = i) &= (M-i)/M, \\
P(X_{n+1} = j|X_n = i) &= 0, \text{ if } j \neq i-1, j \neq i+1.
\end{aligned}$$

Now

$$\begin{aligned}
E(X_{n+1}|X_n = i) &= \sum_{j=0}^{M} jP(X_{n+1} = j|X_n = i) \\
&= (i-1)\frac{i}{M} + (i+1)\left(1 - \frac{i}{M}\right) \\
&= 1 + i\left(1 - \frac{2}{M}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mu_{n+1} = E[E(X_{n+1}|X_n)] &= \sum_{i=0}^{M} E(X_{n+1}|X_n = i)P(X_n = i) \\
&= \sum_{i=0}^{M} P(X_n = i) + \left(1 - \frac{2}{M}\right)\sum_{i=0}^{M} iP(X_n = i) \\
&= 1 + \mu_n\left(1 - \frac{2}{M}\right). \qquad (3)
\end{aligned}$$

To solve the recursion in Eq.(3), subtract $M/2$ from both sides. Then we have

$$\left(\mu_{n+1} - \frac{M}{2}\right) = \left(\mu_n - \frac{M}{2}\right)\left(1 - \frac{2}{M}\right) = \cdots = \left(\mu_0 - \frac{M}{2}\right)\left(1 - \frac{2}{M}\right)^{n+1}.$$