Solution to Final Examination

MAT1300D, Fall 2007

Part I. Multiple-Choice Questions (30 marks)

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
D	E	A	B	C	A	C	E	D	C

1. Find the equation of the tangent line to the graph of $y = 3\sqrt{x} - 1$ when $x = 4$.

(A) $y = \frac{1}{2}x + 1$; (B) $y = \frac{1}{2}x + 3$; (C) $y = \frac{3}{4}x + 7$; **(D)** $y = \frac{3}{4}x + 2$; (E) $y = \frac{3}{4}x - 1$.

Solution. $y' = \frac{3}{2\sqrt{x}}$. When $x = 4$, $y = 5$, and $y' = \frac{3}{4}$. Hence, the equation of the tangent line is $y = \frac{3}{4}(x - 4) + 5$, or $y = \frac{3}{4}x + 2$.

2. Calculate $\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3}$.

(A) $\frac{1}{5}$ (B) 3 (C) $\frac{7}{4}$ (D) $\frac{4}{9}$ **(E)** -7.

Solution. $x^2 - x - 12 = (x + 3)(x - 4)$.

$$\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3} = \lim_{x \to -3} \frac{(x+3)(x-4)}{x+3} = \lim_{x \to -3} (x - 4) = -7.$$

3. On what interval is the function $g(x) = -2x^3 + 12x^2 - 36x + 3$ concave down?

(A) $(2, \infty)$ (B) $(2, 3)$ (C) $(-1, \infty)$ (D) $(-\infty, -1)$ (E) $(-2, 4)$.

Solution. $g' = -6x^2 + 24x - 36$, $g'' = -12x + 24$. $g'' = 0$ implies $x = 2$. When $x < 2$, $g'' <$ 0, the graph of $g(x)$ is concave down.

4. Which of the following statements is true for the function $g(x) = 2x^3 + 3x^2 - 36x + 2$?

(A) $x = -1$ is a local minimum. **(B)** $x = -3$ is a local maximum.
(C) $x = -3$ is a local minimum. (D) $x = 1$ is a local minimum.
(E) $x = 1$ is a local maximum.

Solution. $g' = 6x^2 + 6x - 36$. Critical points are the roots of $x^2 + x - 6 = 0$, $x = -3, 2$. When $-\infty < x < -3$, $g' > 0$; when $-3 < x < 2$, $g' < 0$; when $2 < x < \infty$, $g' > 0$. g attains a local maximum at $x = -3$.

5. Calculate $\int_0^4 (3\sqrt{x}+1)dx$

(A) 10 (B) 15 (C) 20 (D) 25 (E) 30

Solution. $\int_0^4 (3\sqrt{x}+1)dx = 3\int_0^4 x^{1/2}dx + \int_0^4 dx = 3\left(\frac{2}{3}\right)4^{3/2} + 4 = 16 + 4 = 20$.

6a. Suppose that for a certain product, the demand function is given by $D(x) = 11 - x^2$ and the supply function is given by $S(x) = 2x + 3$. Calculate the producer's surplus.

(A) 1; (B) 4; (C) 9; (D) 13; (E) 22.

Solution. The equilibrium point is $11 - x^2 = 2x + 3$, $x = 2$, $p = 7$. The producer's surplus is $\int_0^2 (7-2x-3)dx = \int_0^2 (4-2x)dx = \left[4x - x^2\right]_{x=0}^2 = 4$.

6c. Suppose that for a certain product the demand function is given by $p = \sqrt{180-x}$, $0 \le x \le 180$. For which values of x the demand is elastic?

(A) (0, 120); (B) (120, 180); (C) (0, 90); (D) (90, 180); (E) (90, 120).

Solution. $p/x = \sqrt{180-x}\ /x$, $dp/dx = -\frac{1}{2\sqrt{180-x}}$. $|\eta| = \frac{2(180-x)}{x} > 1$, $360 - 2x > x$, $360 > 3x$, $x < 120$.

7. If $f(x)$ is a function such that $f'(x) = e^{2x}$ and $f(\ln(3)) = 5$, find $f(0)$.

(A) 0; (B) $\frac{1}{2}$; (C) 1; (D) $\frac{5}{2}$; (E) 4.

Solution. $f(x) = \int e^{2x}dx = \frac{1}{2}e^{2x} + C$. $f(\ln(3)) = \frac{1}{2}e^{2\ln(3)} + C = \frac{1}{2}e^{\ln(9)} + C = \frac{9}{2} + C = 5$, $C = \frac{1}{2}$. Hence, $f(0) = \frac{1}{2}e^0 + \frac{1}{2} = 1$.

8. Calculate $\int_1^\infty \frac{1}{\sqrt[3]{x}}dx$.

(A) $\frac{3}{4}$; (B) $\frac{1}{64}$; (C) $\frac{1}{2}$; (D) $\frac{5}{4}$;
(E) This integral is divergent.

Solution. $\int_1^\infty \frac{1}{\sqrt[3]{x}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt[3]{x}}dx = \lim_{b\to\infty}\int_1^b x^{-1/3}dx = \lim_{b\to\infty}\frac{3}{2}\left[b^{2/3}-1\right] = \infty$.

9. If $f(x, y) = x^2y + 3x^3y^2 - 4x$, calculate $f_x(1, 2)$.

(A) −9; (B) 42; (C) −11; **(D)** 36; (E) 27.

Solution. $f_x = 2xy + 9x^2y^2 - 4$. $f_x(1, 2) = 4 + 36 - 4 = 36$.

10. If $f(x, y) = x^2 - 7x + \frac{1}{3}y^3 - 4y + 3$, how many critical points does this function have?

(A) 0; (B) 1; **(C)** 2; (D) 3; (E) 4.

Solution. $f_x = 2x - 7$, $f_y = 3y^2 - 4$. There are two critical points: $\left(\frac{7}{2}, \frac{2}{\sqrt{3}}\right)$ and $\left(\frac{7}{2}, -\frac{2}{\sqrt{3}}\right)$.

Part II. Long Answer Questions (40 marks)

1a. (14 marks) Suppose a population of rabbits grows exponentially. Suppose there are initially 100 rabbits, and after 3 years, there are 160.

(a) How many will there be after 5 years?

(b) How long will it take until there are 200 rabbits?

Solution. $y = 100e^{kt}$. $160 = 100e^{3k}$. $e^{3k} = 8 / 5$. $e^k = \frac{2}{5^{1/3}}$.

(a) $y(5) = 100e^{5k} = 100 \times 2^5 / 5^{5/3}$.

(b) $200 = 100e^{kt} = 100\left(\frac{2}{5^{1/3}}\right)^t$. $\ln 2 = t(\ln 2 - \ln 5 / 3)$. $t = \ln 2 / (\ln 2 - \ln 5 / 3)$.

1b. (14 marks) Suppose a population of rabbits grows exponentially. Suppose there are initially 100 rabbits, and after 4 years, there are 180.

(a) How many will there be after 6 years?

(b) How long will it take until there are 300 rabbits?

Solution. $y = 100e^{kt}$. $180 = 100e^{4k}$. $e^{4k} = 9 / 5$. $e^k = \dfrac{3^{1/2}}{5^{1/4}}$.

(a) $y(6) = 100e^{6k} = 100 \times 3^3 / 5^{3/2}$.

(b) $300 = 100e^{kt} = 100\left(\dfrac{3^{1/2}}{5^{1/4}}\right)^t$. $\ln 3 = t\,(\ln 3 / 2 - \ln 5 / 4)$. $t = \ln 3 / (\ln 3 / 2 - \ln 5 / 4)$.

2a. (14 marks) Calculate the following indefinite integrals:

(a) $\int xe^{4x}dx$.

Solution.

$$\int xe^{4x}dx = \frac{1}{4}\int xde^{4x} = \frac{1}{4}\left(xe^{4x} - \int e^{4x}dx\right) = \frac{1}{4}\left(xe^{4x} - \frac{1}{4}e^{4x}\right) + C = \frac{e^{4x}}{16}(4x-1) + C.$$

(b) $\int \dfrac{\ln(x)}{x}dx$.

Solution. Let $u = \ln x$. $\int \dfrac{\ln(x)}{x}dx = \int udu = \dfrac{u^2}{2} + C = \dfrac{1}{2}(\ln x)^2 + C$.

2b. (14 marks) Calculate the following indefinite integrals:

(a) $\int xe^{5x}dx$.

Solution.

$$\int xe^{5x}dx = \frac{1}{5}\int xde^{5x} = \frac{1}{5}\left(xe^{5x} - \int e^{5x}dx\right) = \frac{1}{5}\left(xe^{5x} - \frac{1}{5}e^{5x}\right) + C = \frac{e^{5x}}{25}(5x-1) + C.$$

(b) $\int \dfrac{\ln(x)}{x}dx$.

Solution. Let $u = \ln x$. $\int \frac{\ln(x)}{x}dx = \int u du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C$.

3a. (14 marks) Suppose a farmer wishes to enclose a rectangular region against an existing wall using fencing. (No fencing is required against the wall). The sides perpendicular to the wall cost \$30 per foot, and the side parallel to the wall cost \$40 per foot. The farmer has \$1,200 to spend. What dimensions maximize the area of the region? Be sure to explain why your answer is an absolute maximum.

Solution. Let the length of side perpendicular to the wall be x and let the length of the side parallel to the wall be y. Then the area enclosed is $A = xy$. The total cost is $60x + 40y = 1200$, or $3x + 2y = 60$. $y = 30 - 3x / 2$. Then $A = 30x - 3x^2 / 2$. $0 \le x \le 20$.

$A' = 30 - 3x$. $x = 10$ is a critical point. When $x < 10$, $A' > 0$, A is increasing; when $x < 10$, $A' < 0$, A is decreasing. A attains a local maximum at $x = 10$. Since, $A(0) = A(20) = 0$. This local maximum is an absolute maximum. Then $y = 15$.

3b. (14 marks) Suppose a farmer wishes to enclose a rectangular region against an existing wall using fencing. (No fencing is required against the wall). The sides perpendicular to the wall cost \$50 per foot, and the side parallel to the wall cost \$40 per foot. The farmer has \$1,600 to spend. What dimensions maximize the area of the region? Be sure to explain why your answer is an absolute maximum.

Solution. Let the length of side perpendicular to the wall be x and let the length of the side parallel to the wall be y. Then the area enclosed is $A = xy$. The total cost is $100x + 40y = 1600$, or $5x + 2y = 80$. $y = 40 - 5x / 2$. Then $A = 40x - 5x^2 / 2$. $0 \le x \le 16$.

$A' = 40 - 5x$. $x = 8$ is a critical point. When $x < 8$, $A' > 0$, A is increasing; when $x < 8$, $A' < 0$, A is decreasing. A attains a local maximum at $x = 8$. $y = 20$. Since, $A(0) = A(16) = 0$. This local maximum is an absolute maximum.

4a. (16 marks) Consider two functions $y = 4 - \frac{1}{4}x^2$, and $y = x + 1$.

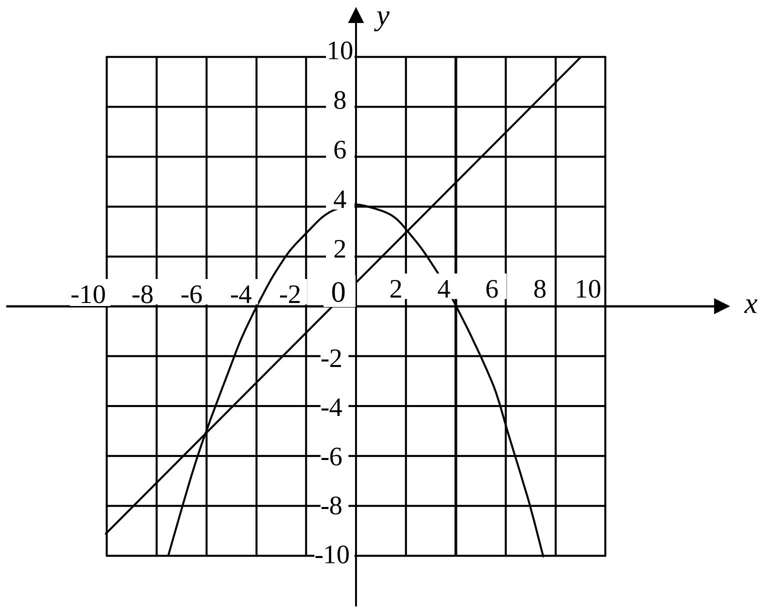
(a) (4 points) Find the intersection points of the graphs of these two functions.

(b) (6 points) On the next page, graph these functions, and shade the region bounded by the graphs of these two functions and lines $x = 0$, and $x = 4$.

(c) (6 points) Find the area of the shaded region.

Solution. (a) $4 - \frac{1}{4}x^2 = x + 1$, $x^2 + 4x - 12 = 0$. $x = -6, 2$. The intersection points are $(-6, -5)$ and $(2, 3)$.

(b)



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(c) The area of the bounded region is

$$A = \int_0^2 \left(\left(4 - \frac{1}{4}x^2\right) - (x+1)\right)dx + \int_2^4 \left((x+1) - \left(4 - \frac{1}{4}x^2\right)\right)dx$$

$$\int_0^2 \left(3 - \frac{1}{4}x^2 - x\right)dx + \int_2^4 \left(3 - \frac{1}{4}x^2 - x\right)dx = \left[3x - \frac{x^3}{12} - \frac{x^2}{2}\right]_{x=0}^{2} + \left[-3x + \frac{x^3}{12} + \frac{x^2}{2}\right]_{x=2}^{4} = 8.$$

4b. (16 marks) Consider two functions $y = 4 - \frac{1}{4}x^2$, and $y = 2x - 1$.

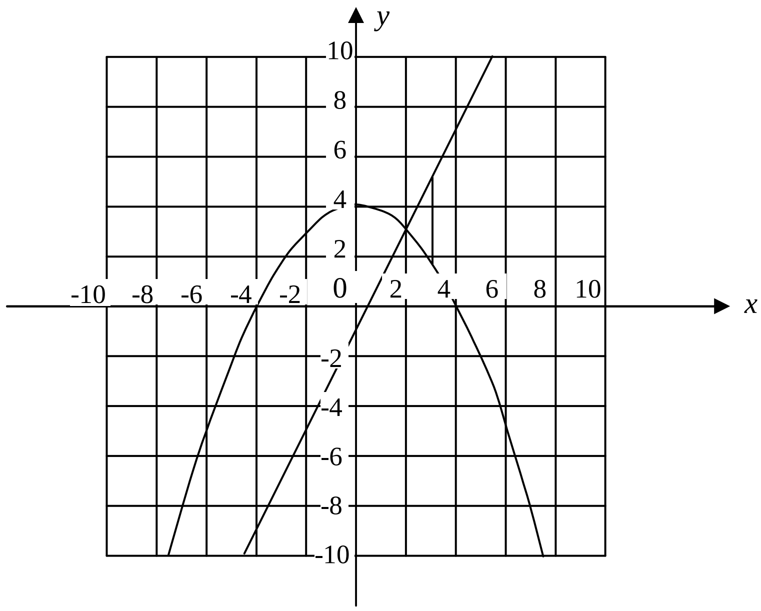
(a) (4 points) Find the intersection points of the graphs of these two functions.

(b) (6 points) On the next page, graph these functions, and shade the region bounded by the graphs of these two functions and lines $x = 0$, and $x = 3$.

(c) (6 points) Find the area of the shaded region.

Solution. (a) $4 - \frac{1}{4}x^2 = 2x - 1$, $x^2 + 8x - 20 = 0$. $x = -10, 2$. The intersection points are $(-10, -21)$ and $(2, 1)$.

(b)



(c) The area of the bounded region is

$$A = \int_0^2 \left(4 - \frac{1}{4}x^2 - 2x + 1\right)dx + \int_2^3 \left(2x - 1 - 4 + \frac{1}{4}x^2\right)dx$$

$$\int_0^2 \left(5 - \frac{x^2}{4} - 2x\right)dx + \int_2^3 \left(-5 + \frac{x^2}{4} + 2x\right)dx = \left[5x - \frac{x^3}{12} - x^2\right]_{x=0}^{2} + \left[-5x + \frac{x^3}{12} + x^2\right]_{x=2}^{3} = \frac{83}{12} .$$

5a. (12 marks) Consider the function of two variables

$$f(x, y) = 2x^2 - 4xy + 4y^2 - 5x + 9y + 3.$$

(a) Calculate the first-order partial derivatives.

(b) Find all critical points.

(c) Identify what type of critical points they are (local maximum, local minimum or saddle point).

Solution. (a) $f_x = 4x - 4y - 5$, $f_y = -4x + 8y + 9$.

(b) $y = -1$, $x = 1 / 4$.

(c) $f_{xx} = 4$, $f_{yy} = 8$, $f_{xy} = -4$, $D = f_{xx}f_{yy} - f_{xy}^{\ 2} = 16 > 0$, and $f_{xx} > 0$, the critical point $(1 / 4, -1)$ corresponds to a local minimum.

5b. (12 marks) Consider the function of two variables

$$f(x, y) = 2x^2 - 4xy + 4y^2 - 5x + 13y + 3.$$

(a) Calculate the first-order partial derivatives.

(b) Find all critical points.

(c) Identify what type of critical points they are (local maximum, local minimum or saddle point).

Solution. (a) $f_x = 4x - 4y - 5$, $f_y = -4x + 8y + 13$.

(b) $y = -2$, $x = -3 / 4$.

(c) $f_{xx} = 4$, $f_{yy} = 8$, $f_{xy} = -4$, $D = f_{xx}f_{yy} - f_{xy}^2 = 16 > 0$, and $f_{xx} > 0$, the critical point $(1 / 4, -1)$ corresponds to a local minimum.