

Determinants

Recall that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we have $\det A = ad - bc$.

A is invertible $\iff \det A \neq 0$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For an $n \times n$ matrix we use

$$\det A = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| - a_{14}|A_{14}| + \dots$$

where we continue alternating signs.

Definition: Let A be an $n \times n$ square matrix, and denote the entries of A by a_{ij} (row i , column j). Let A_{ij} denote the submatrix formed by deleting row i and column j from A . Then

$$\begin{aligned} \det A &= (-1)^{1+1}a_{11} \det A_{11} \\ &\quad + (-1)^{1+2}a_{12} \det A_{12} \\ &\quad + \dots \\ &\quad + (-1)^{1+n}a_{1n} \det A_{1n} \end{aligned}$$

That is,

$$\det A = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}$$

This is called **cofactor expansion** along the first row.

Example 1: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$. Find $\det A$.

Solution:

$$A_{11} = \begin{bmatrix} 5 & 6 \\ -8 & 9 \end{bmatrix}, A_{12} = \begin{bmatrix} -4 & 6 \\ 7 & 9 \end{bmatrix}, A_{13} = \begin{bmatrix} -4 & 5 \\ 7 & -8 \end{bmatrix}$$

$$\det A_{11} = (5)(9) - (-8)(6) = 45 + 48 = 93$$

$$\det A_{12} = (-4)(9) - (7)(6) = -36 - 42 = -78$$

$$\det A_{13} = (-4)(-8) - (7)(5) = 32 - 35 = -3$$

$$\begin{aligned} \det A &= (-1)^{1+1}a_{11} \det A_{11} \\ &\quad + (-1)^{1+2}a_{12} \det A_{12} \\ &\quad + (-1)^{1+3}a_{13} \det A_{13} \\ &= (-1)^2 a_{11} \det A_{11} \\ &\quad + (-1)^3 a_{12} \det A_{12} \\ &\quad + (-1)^4 a_{13} \det A_{13} \\ &= (1)(1)(93) + (-1)(2)(-78) + (1)(3)(-3) \\ &= 93 + 156 - 9 \\ &= 240 \end{aligned}$$

We can actually use the same pattern of formula to calculate the determinant of A by expanding along the **second row**:

$$A_{21} = \begin{bmatrix} 2 & 3 \\ -8 & 9 \end{bmatrix}, A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}, A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & -8 \end{bmatrix}$$

$$\det A_{21} = (2)(9) - (-8)(3) = 18 + 24 = 42$$

$$\det A_{22} = (1)(9) - (7)(3) = 9 - 21 = -12$$

$$\det A_{23} = (1)(-8) - (7)(2) = -8 - 14 = -22$$

$$\begin{aligned} \det A &= (-1)^{2+1}a_{21} \det A_{21} \\ &\quad + (-1)^{2+2}a_{22} \det A_{22} \\ &\quad + (-1)^{2+3}a_{23} \det A_{23} \\ &= (-1)^3 a_{21} \det A_{21} \\ &\quad + (-1)^4 a_{22} \det A_{22} \\ &\quad + (-1)^5 a_{23} \det A_{23} \\ &= -a_{21} \det A_{21} + a_{22} \det A_{22} - a_{23} \det A_{23} \\ &= -(-4)(42) + (5)(-12) - (6)(-22) \\ &= 168 - 60 + 132 \\ &= 240 \end{aligned}$$

Similarly we can CAREFULLY adapt our formula for the determinant of A in order to calculate the determinant along ANY row or column.

Formula for determinant by expanding along row 1:

$$\begin{aligned}\det A &= (-1)^{1+1}a_{11} \det A_{11} \\ &\quad + (-1)^{1+2}a_{12} \det A_{12} \\ &\quad + \dots \\ &\quad + (-1)^{1+n}a_{1n} \det A_{1n}\end{aligned}$$

Example: Find $\det A$ if $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 4 & 1 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}$

Solution:

Example: Find $\det A$ if $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ -5 & -8 & 4 & -3 \end{bmatrix}$

Solution:

Properties of Determinants

Let A and B be two $n \times n$ matrices. Then

(a) $\det A = \det A^T$

(b) $\det(AB) = \det A \det B$

(c) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$

Proof of (c): Since A is invertible,

$$\begin{aligned} AA^{-1} &= I \\ \implies \det(AA^{-1}) &= \det I = 1 \\ \implies \det A \cdot \det A^{-1} &= 1 \\ \implies \det A^{-1} &= \frac{1}{\det A} \end{aligned}$$

Note: In general, $\det(A + B) \neq \det A + \det B$

For example, let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Then

$$\det A = (1)(5) - (2)(2) = 5 - 4 = 1$$

$$\det B = (3)(3) - (1)(1) = 9 - 1 = 8$$

$$\det A + \det B = 1 + 8 = 9$$

$$A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

$$\det(A + B) = (4)(8) - (3)(3) = 32 - 9 = 23$$

Hence $\det(A + B) \neq \det A + \det B$.

Let A be an $n \times n$ matrix.

The **definition** of the determinant of A is

$$\det A = (-1)^{1+1}a_{11}|A_{11}| + (-1)^{1+2}a_{12}|A_{12}| + \cdots + (-1)^{1+n}a_{1n}|A_{1n}|$$

The determinant of A can be calculated by CAREFULLY altering the formula to expand along any row or column:

Expand along row i instead

$$\det A = (-1)^{i+1}a_{i1}|A_{i1}| + (-1)^{i+2}a_{i2}|A_{i2}| + \cdots + (-1)^{i+n}a_{in}|A_{in}|$$

Expand down column j instead

$$\det A = (-1)^{1+j}a_{1j}|A_{1j}| + (-1)^{2+j}a_{2j}|A_{2j}| + \cdots + (-1)^{n+j}a_{nj}|A_{nj}|$$

Theorem: Let A be an $n \times n$ matrix. Then

$$A \text{ is invertible} \iff \det A \neq 0$$

This theorem can be added to the invertible matrix theorem.

The effect of row operations on determinants

Let A be an $n \times n$ square matrix.

(i) If B is obtained from A by interchanging two different rows, then

$$\det B = -\det A$$

(ii) If B is obtained from A by multiplying a row by a scalar k , then

$$\det B = k \det A$$

(iii) If B is obtained from A by adding a multiple of one row of A to another row, then

$$\det B = \det A$$

Note: Column operations have the same effect on determinants as row operations

Examples:

If $A \sim B$ where performing $R_1 \longleftrightarrow R_2$ on A gives B ,
then $\det B = -\det A$

If $A \sim B$ where performing $R'_1 = 5R_1$ on A gives B ,
then $\det B = 5 \det A$

If $A \sim B$ where performing $R'_1 = R_1 + 7R_2$ on A gives B ,
then $\det B = \det A$

Remarks:

1) Let A be an $n \times n$ matrix.

(a) If A has two identical rows (or columns), then

$$\det A = 0$$

(b) If A has a zero row (or zero column), then

$$\det A = 0$$

2) Let I_n be the $n \times n$ identity matrix, and let E be an $n \times n$ elementary matrix.

(a) If E results from interchanging two rows of I_n , then

$$\det E = -1$$

(b) If E results from multiplying a row of I_n by k , then

$$\det E = k$$

(c) If E results from adding a multiple of one row of I_n to another row, then

$$\det E = 1$$

Examples:

$$\begin{vmatrix} 1 & 3 & 2 & 3 \\ 4 & 4 & 5 & 4 \\ 5 & 1 & -3 & 1 \\ -7 & 2 & 8 & 2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 5 & -2 \\ 5 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Take I_4 and perform $R'_2 = 5R_2$ to get E_1 . Then

$$|E_1| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 5 \quad (= 5 \det I_4)$$

Take I_4 and perform $R_3 \longleftrightarrow R_4$ to get E_2 . Then

$$|E_2| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \quad (= -1 \det I_4)$$

Take I_4 and perform $R'_2 = R_2 + 3R_4$ to get E_3 . Then

$$|E_3| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \quad (= \det I_4)$$

Example: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 2 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

Find $\det A$ by row reducing A .

Solution:

$\det A$

$$\begin{aligned} &= \begin{vmatrix} 1 & 2 & -1 \\ 5 & 2 & 5 \\ -2 & 0 & 1 \end{vmatrix} \begin{array}{l} R'_2 = R_2 + (-5)R_1 \text{ (doesn't change det)} \\ R'_3 = R_3 + 2R_1 \text{ (doesn't change det)} \end{array} \\ &= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -8 & 10 \\ 0 & 4 & -1 \end{vmatrix} \begin{array}{l} R'_3 = R_3 + (1/2)R_2 \text{ (doesn't change det)} \end{array} \\ &= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -8 & 10 \\ 0 & 0 & 4 \end{vmatrix} \begin{array}{l} \text{determinant of a triangular matrix} \\ \text{is product of diagonal entries} \end{array} \\ &= (1)(-8)(4) \\ &= -32 \end{aligned}$$

Example: Let $A = \begin{bmatrix} 1 & 2 & -1 & -1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}$. Find $\det A$.

Solution: We will use row operations to simplify Column 2, and then expand down column 2.

$$\begin{aligned}
 \det A &= \begin{vmatrix} 1 & 2 & -1 & -1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} && R'_1 = R_1 + 2R_3 \text{ (doesn't change det)} \\
 &= \begin{vmatrix} 7 & 0 & 1 & 3 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} && \text{(now expand using Column 2)} \\
 &= (-1)^{3+2}(-1) \begin{vmatrix} 7 & 1 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix} \\
 &= \begin{vmatrix} 7 & 1 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix} && \begin{matrix} R'_2 = R_2 + (-2)R_1 \\ R'_3 = R_3 + R_1 \end{matrix} \left(\begin{matrix} \text{Simplify} \\ 2^{\text{nd}} \text{ column} \end{matrix} \right) \\
 &= \begin{vmatrix} 7 & 1 & 3 \\ -15 & 0 & -8 \\ 9 & 0 & 5 \end{vmatrix} \\
 &= (-1)^{1+2}(1) \begin{vmatrix} -15 & -8 \\ 9 & 5 \end{vmatrix} \\
 &= (-1) [(-15)(5) - (9)(-8)] \\
 &= (-1)(-75 + 72) \\
 &= (-1)(-3) \\
 &= 3
 \end{aligned}$$

Example: Let $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -3$. Find

$$\begin{vmatrix} a_1 + b_1 & a_1 - b_1 & c_1 \\ a_2 + b_2 & a_2 - b_2 & c_2 \\ a_3 + b_3 & a_3 - b_3 & c_3 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} a_1 + b_1 & a_1 - b_1 & c_1 \\ a_2 + b_2 & a_2 - b_2 & c_2 \\ a_3 + b_3 & a_3 - b_3 & c_3 \end{vmatrix} \quad C'_2 = C_2 + C_1$$

$$= \begin{vmatrix} a_1 + b_1 & 2a_1 & c_1 \\ a_2 + b_2 & 2a_2 & c_2 \\ a_3 + b_3 & 2a_3 & c_3 \end{vmatrix}$$

$$= 2 \begin{vmatrix} a_1 + b_1 & a_1 & c_1 \\ a_2 + b_2 & a_2 & c_2 \\ a_3 + b_3 & a_3 & c_3 \end{vmatrix} \quad C'_1 = C_1 + (-1)C_2$$

$$= 2 \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \quad C_1 \longleftrightarrow C_2$$

$$= 2(-1) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= (-2)(-3)$$

$$= 6$$

Exercise: Let $A = \begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix}$ and $\det A = -6$.

Evaluate the determinants of the following matrices:

(i) $A^{-1} A^T A$

(ii) $2A$

(iii) $\begin{bmatrix} a & b & 2a + c \\ u & v & 2u + w \\ x & y & 2x + z \end{bmatrix}$

(iv) $\begin{bmatrix} 3a & 3b & 3c \\ 4x & 4y & 4z \\ -u & -v & -w \end{bmatrix}$

Solutions:

(i)

$$\begin{aligned}\det(A^{-1} A^T A) &= (\det A^{-1}) (\det A^T) \det A \\ &= \frac{1}{\det A} (\det A)(\det A) \\ &= \frac{1}{-6}(-6)(-6) = -6\end{aligned}$$

(ii)

$$\begin{aligned}\det(2A) &= (2)(2)(2) \det A \quad (\text{one for each row!}) \\ &= 2^3 \det A \\ &= 8(-6) = -48\end{aligned}$$

(iii)

$$\begin{aligned}\begin{vmatrix} a & b & 2a + c \\ u & v & 2u + w \\ x & y & 2x + z \end{vmatrix} & C'_3 = C_3 + (-2)C_1 \\ &= \begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix} \\ &= \det A = -6\end{aligned}$$

(iv)

$$\begin{aligned} & \begin{vmatrix} 3a & 3b & 3c \\ 4x & 4y & 4z \\ -u & -v & -w \end{vmatrix} R'_1 = \frac{1}{3}R_1 \\ = & (3) \begin{vmatrix} a & b & c \\ 4x & 4y & 4z \\ -u & -v & -w \end{vmatrix} R'_2 = \frac{1}{4}R_2 \\ = & (3)(4) \begin{vmatrix} a & b & c \\ x & y & z \\ -u & -v & -w \end{vmatrix} R'_3 = -R_3 \\ = & (3)(4)(-1) \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} R_2 \longleftrightarrow R_3 \\ = & (3)(4)(-1) \begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix} \\ = & (3)(4)(-1)(-6) \\ = & -72 \end{aligned}$$

Example: If A and B are 5×5 matrices and

$$\det(-2A^T) = 128 = \det(A^3(B^T)^{-1})$$

find $\det A$ and $\det B$.

Solution: First we use the equation $\det(-2A^T) = 128$ to solve for $\det A$:

$$\begin{aligned}\det(-2A^T) &= (-2)^5 \det(A^T) \\ &= -32 \det A \\ &= 128 \\ \implies \det A &= \frac{128}{-32} = -4\end{aligned}$$

Now we use the equations $\det(A^3(B^T)^{-1}) = 128$ and $\det A = -4$ to solve for $\det B$:

$$\begin{aligned}\det(A^3(B^T)^{-1}) &= \det(A^3) \det((B^T)^{-1}) \\ &= (\det A)^3 \det((B^{-1})^T) \\ &= (\det A)^3 \det(B^{-1}) \\ &= (-4)^3 \frac{1}{\det B} \\ &= 128 \\ \implies \det B &= \frac{(-4)^3}{128} = -\frac{1}{2}\end{aligned}$$

Hence $\det A = -4$ and $\det B = -\frac{1}{2}$.

Example: Show that if A is an $n \times n$ matrix, then

$$\det(kA) = k^n \det A.$$

Solution:

$$\begin{aligned} kA &= (kI)A \\ \det(kA) &= \det((kI)A) \\ &= \det(kI) \det A \\ &= k^n \det A \end{aligned}$$

Exercise 1: Suppose that

$$\det A = -3, \quad \det B = 2, \quad \det C = 5$$

Compute

$$\det(A^2BC^{-1}B^T)$$

Exercise 2: Show that

$$\begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$$

has an inverse for any values of $a, b,$ and $c.$