CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE & SOFTWARE ENGINEERING

COMP 232/4 **Mathematics for Computer Science** Winter 2014

Assignment 2 Due date: February 24, 2014

1. For each of the arguments below, formalize them in propositional logic. If the argument is valid identify which inference rule was used, and formulate the tautology underlying the rule. If the argument is invalid, state whether the inverse or converse error was made.

   (a) All cheaters sit in the back row.
   George sits in the back row.
   ∴ George is a cheater.

   Solution: Invalid.

   The argument is of the form
   Cheater(George) → Backrow(George)
   Backrow(George)
   ∴ Cheater(Georg)

   It is an example of the converse error.

   (b) For all students x, if x studies discrete math, then x is good at logic.
   Helen studies discrete math.
   ∴ Helen is good at logic.

   Solution: Valid.

   The argument is of the form
   Studiesmath(Helen) → Goodatlogic(Helen)
   Studiesmath(Helen)
   ∴ Goodatlogic(Helen)

   It is an example of Modus Ponens.

(c) If the compilation of a computer program produces error messages, then the program is not correct or the compiler is faulty.
The compilation of this program does not produce error messages.
∴ this program is correct and the compiler is not faulty.

Solution: Invalid.

The argument is of the form

$\mathrm{CE(p)} \rightarrow \Big(\mathrm{IP(p)} \vee \mathrm{FC(p)}\Big)$

$\neg\ \mathrm{CE(p)}$

$\therefore \neg\Big(\ \mathrm{IP(p)} \vee \mathrm{FC(p)}\Big)$

It is an example of an inverse error

(d) All students who do not do their homework and do not study the course material will not get a good course grade.
John gets a good course grade.
∴ John did his homework or studied the course material.

Solution: Valid.

The argument is of the form

$\Big(\neg\mathrm{Homework(John)} \wedge\neg\ \mathrm{Study(John)}\Big) \rightarrow \neg\ \mathrm{Goodgrade(John)}$

$\mathrm{Goodgrade(John)}$

$\therefore\ \mathrm{Homework(John)} \vee \mathrm{Study(John)}$

It is an example of Modus Tollens (followed by De Morgan)

2. For each of the premise-conclusion pairs below, give a valid step-by-step argument (proof) along with the name of the inference rule used in each step. For examples, see pages 73 and 74 in textbook.

   (a) Premise: $\{\neg p \vee q \rightarrow r,\ s \vee \neg q,\ \neg t,\ p \rightarrow t,\ \neg p \wedge r \rightarrow \neg s\}$, conclusion: $\neg q$.

   Solution:

Step	**Conclusion**	**Reason**
1.	$p \rightarrow t$	Premise
2.	$\neg t$	Premise
3.	$\neg p$	Modus Tollens using (1) and (2)
4.	$\neg p \vee q$	Addition using (3)
5.	$\neg p \vee q \rightarrow r$	Premise
6.	r	Modus Ponens using (4) and (5)
7.	$\neg p \wedge r$	Conjunction using (3) and (6)
8.	$\neg p \wedge r \rightarrow \neg s$	Premise
9.	$\neg s$	Modus Ponens using (7) and (8)
10.	$s \vee \neg q$	Premise
11.	$\neg q$	Disjunctive Syllogism using (9) and (10)

   (b) Premise: $\{\neg p \rightarrow r \wedge \neg s,\ t \rightarrow s,\ u \rightarrow \neg p,\ \neg w,\ u \vee w\}$, conclusion: $\neg t \vee w$.

   Solution:

Step	**Conclusion**	**Reason**
1.	$u \vee w$	Premise
2.	$\neg w$	Premise
3.	u	Disjunctive Syllogism using (1) and (2)
4.	$u \rightarrow \neg p$	Premise
5.	$\neg p$	Modus Ponens using (3) and (4)
6.	$\neg p \rightarrow r \wedge \neg s$	Premise
7.	$r \wedge \neg s$	Modus Ponens using (5) and (6)
8.	$\neg s$	Simplification using (7)
9.	$t \rightarrow s$	Premise
10.	$\neg t$	Modus Tollens using (8) and (9)
11.	$\neg t \vee w$	Addition using (10)

(c) Premise: $\{p \vee q,\ q \to r,\ p \wedge s \to t,\ \neg r,\ \neg q \to u \wedge s\}$, conclusion: t.

Solution:

Step	**Conclusion**	**Reason**
1.	$\neg r$	Premise
2.	$q \to r$	Premise
3.	$\neg q$	Modus Tollens using (1) and (2)
4.	$\neg q \to u \wedge s$	Premise
5.	$u \wedge s$	Modus Ponens using (3) and (4)
6.	s	Simplification using (5)
7.	$p \vee q$	Premise
8.	p	Disjunctive Syllogism using (3) and (7)
9.	$p \wedge s$	Conjunction using (6) and (8)
10.	$p \wedge s \to t$	Premise
11.	t	Modus Ponens using (9) and (10)

3. Use rules of inference to show that

   (a) $\forall x \Big(R(x) \to \Big(S(x) \vee Q(x)\Big)\Big)$

   $\exists x \Big(\neg S(x)\Big)$

   $\therefore \exists x \Big(R(x) \to Q(x)\Big)$

Solution:

(1)	$\exists x \Big(\neg S(x)\Big)$	Premise
(2)	$\neg S(c)$	Existential instantiation from (1)
(3)	$\forall x \Big(R(x) \to \Big(S(x) \vee Q(x)\Big)\Big)$	Premise
(4)	$R(c) \to \Big(S(c) \vee Q(c)\Big)$	Universal instantiation from (3)
(5)	$\neg R(c) \vee \Big(S(c) \vee Q(c)\Big)$	$p \to q \equiv \neg p \vee q$ from (4)
(6)	$\Big(\neg R(c) \vee S(c)\Big) \vee Q(c)$	Associativity from (5)
(7)	$\Big(S(c) \vee \neg R(c)\Big) \vee Q(c)$	Commutativity from (6)
(8)	$S(c) \vee \Big(\neg R(c) \vee Q(c)\Big)$	Associativity from (7)
(9)	$\neg R(c) \vee Q(c)$	Disjunctive syllogism from (8) and (2)
(10)	$R(c) \to Q(c)$	$p \to q \equiv \neg p \vee q$ from (9)
(11)	$\exists x \Big(R(x) \to Q(x)\Big)$	Existential generalization from (10)

(b) $$\begin{array}{l} \forall x\Big(P(x)\lor Q(x)\Big)\\ \forall x\Big(\Big(\neg P(x)\land Q(x)\Big)\to R(x)\Big)\\ \hline \forall x\Big(\neg R(x)\to P(x)\Big)\end{array}$$

Solution:

(1)	$\forall x\Big(P(x)\lor Q(x)\Big)$	Premise
(2)	$P(c)\lor Q(c)$	Universal Instantiation from (1)
(3)	$\forall x\Big(\Big(\neg P(x)\land Q(x)\Big)\to R(x)\Big)$	Premise
(4)	$\Big(\neg P(c)\land Q(c)\Big)\to R(c)$	Universal Instantiation from (3)
(5)	$\neg\Big(\neg P(c)\land Q(c)\Big)\lor R(c)$	$p\to q\equiv\ \neg p\lor q$ from (4)
(6)	$\Big(\neg\neg P(c)\lor\neg Q(c)\Big)\lor R(c)$	De Morgan from (5)
(6)	$\Big(P(c)\lor\neg Q(c)\Big)\lor R(c)$	Double negation from (6)
(7)	$\Big(P(c)\lor R(c)\Big)\lor\neg Q(c)$	Commutativity and Associativity fro m (6)
(8)	$P(c)\lor\Big(P(c)\lor R(c)\Big)$	Resolution from (2) and (7)
(9)	$\Big(P(c)\lor P(c)\Big)\lor R(c)$	Associativity from (8)
(10)	$P(c)\lor R(c)$	Idempotency from (9)
(11)	$R(c)\lor P(c)$	Commutativity from (10)
(12)	$\neg R(c)\to P(c)$	$p\to q\equiv\neg p\lor q$ from (11)
(13)	$\forall x\Big(\neg R(x)\to P(x)\Big)$	Universal generalization from (11)

4. Prove that the following four statements are equivalent:

   (a) n^2 is odd.

   (b) $1 - n$ is even.

   (c) n^3 is odd.

   (d) $n^2 + 1$ is even.

Solution:

We will show that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$. The claim then follows (why?).

- $(a) \Rightarrow (b)$. We show the contrapositive $odd(1-n) \Rightarrow even(n^2)$.

  $odd(1-n) \Rightarrow 1 - n = 2k + 1$, for some $k \in \mathbb{Z}$
  $\Rightarrow n = -2k$
  $\Rightarrow n = 2(-k)$
  $\Rightarrow n^2 = \Big(2(-k)\Big)^2 = 2\Big(2k^2\Big)$
  $\Rightarrow even(n^2)$.

- $(b) \Rightarrow (c)$.
  $even(1-n) \Rightarrow 1 - n = 2k$, for some $k \in \mathbb{Z}$
  $\Rightarrow -n = 2k - 1$
  $\Rightarrow n = -2k + 1$
  $\Rightarrow n^3 = \Big(-2k + 1\Big)^3$
  $\Rightarrow n^3 = -8k^3 + 12k^2 - 6k + 1$
  $\Rightarrow n^3 = 2\Big(-4k^3 + 6k^2 - 3k\Big) + 1$
  $\Rightarrow odd(n^3)$.

- $(c) \Rightarrow (d)$. We show the contrapositive $odd(n^2+1) \Rightarrow even(n^3)$.
  $odd(n^2 + 1) \Rightarrow even(n^2)$
  $\Rightarrow n^2 = 2k$, for some $k \in \mathbb{Z}$
  $\Rightarrow n^3 = n \cdot n^2 = n(2k) = 2(nk)$
  $\Rightarrow even(n^3)$.

- $(d) \Rightarrow (a)$.
  $even(n^2 + 1) \Rightarrow n^2 + 1 = 2k$, for some $k \in \mathbb{Z}$
  $\Rightarrow n^2 = 2k - 1 = 2k - 1 + 1 - 1$
  $\Rightarrow n^2 = 2k - 2 + 1 = 2(k-1) + 1$
  $\Rightarrow odd(n^2)$.

5. (a) Give a direct proof of: "If x is an odd integer and y is an even integer, then $x + y$ is odd."

Solution:
$odd(x) \Rightarrow x = 2k + 1$, for some $k \in \mathbb{Z}$
$even(y) \Rightarrow y = 2k'$, for some $k' \in \mathbb{Z}$
$\Rightarrow x + y = 2k + 1 + 2k' = 2(k + k') + 1$
$\Rightarrow odd(x + y)$.

(b) Give a proof by contradiction of: "If n is an odd integer, then n^2 is odd."

Solution: Suppose to the contrary that $odd(n)$ and $even(n^2)$.

$odd(n) \Rightarrow n = 2k + 1$, for some $k \in \mathbb{Z}$
$\Rightarrow n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
$\Rightarrow odd(n^2)$, which contradicts $even(n^2)$.

(c) Give an indirect proof of: "If n is an odd integer, then $n + 2$ is odd."

Solution: Using a proof by contrapositive, we need to prove $even(n + 2) \Rightarrow even(n)$.

$even(n + 2) \Rightarrow n + 2 = 2k$, for some $k \in \mathbb{Z}$
$\Rightarrow n = 2k - 2 = 2(k - 1) \Rightarrow even(n)$.

6. Is the statement "For all positive $x, y \in \mathbb{R}$, if x is irrational and y is irrational then $x + y$ is irrational" True of False? If True then give a proof. If False then explain why, e.g., by giving a counterexample.

Solution:

False: Counterexample: Let $x_1 = \frac{7}{4} - \sqrt{2}$ and $x_2 = \frac{7}{4} + \sqrt{2}$. Then both x_1 and x_2 are positive and irrational, while their sum is $\frac{7}{2}$, which is rational.

For completeness we must also demonstrate that x_1 and x_2 are indeed irrational. This is most easily done by contradiction: Suppose $x_1 = \frac{7}{4} - \sqrt{2}$ is rational. Then $x_1 = \frac{7}{4} - \sqrt{2} = \frac{p}{q}$, for positive integers p and q. It follows that $\sqrt{2} = \frac{7}{4} - \frac{p}{q} = \frac{7q-4p}{4q}$, which is rational. But $\sqrt{2}$ is known to be irrational, and thus we have a contradiction. The proof that x_2 is irrational is very similar to that for x_1. (The fact that $\sqrt{2}$ is irrational is demonstrated in the textbook.)

7. Consider the statement concerning integers "*If* $m+n$ *is even, then* $m-n$ *is even.*"

   (a) Give a direct proof of the statement.

   Solution:
   $even(m+n) \Rightarrow m+n = 2k$, for some $k \in \mathbb{Z}$
   $\Rightarrow n = (2k-m)$
   $\Rightarrow m-n = m-2k+m = 2m-2k = 2(m-k)$
   $\Rightarrow even(m-n)$.

   (b) Give an indirect proof of the statement.

   Solution: We need to prove $odd(m-n) \Rightarrow odd(m+n)$.

   $odd(m-n) \Rightarrow m-n = 2k+1$, for some $k \in \mathbb{Z}$
   $\Rightarrow m = 2k+1+n$
   $\Rightarrow m+n = 2k+1+n+n = 2(k+n)+1$
   $\Rightarrow odd(m+n)$.

   (c) Prove the statement by contradiction.

   Solution: We need to prove that $even(m+n) \wedge odd(m-n)$ leads to a contradiction.

   $odd(m-n) \Rightarrow m-n = 2k+1$, for some $k \in \mathbb{Z}$
   $\Rightarrow m = 2k+1+n$
   $\Rightarrow m+n = 2k+1+n+n = 2(k+n)+1$
   $\Rightarrow odd(m+n)$, which contradicts $even(m+n)$.

   Therefore, the given statement is true, that is, $even(m+n) \Rightarrow even(m-n)$.

   Comment on Question 7 on the next page

Comment on Question 7: For the particular statement in question 7 the proof by contraposition and the proof by contradiction are essentially the same.

However, in a proof by contraposition of $P \Rightarrow Q$ we prove $\neg Q \Rightarrow \neg P$. In a proof by contradiction of $P \Rightarrow Q$ we assume P and $\neg Q$ and derive *some* contradiction. In question 7 (c) the contradiction happens to be $\neg P$.

Here is an example where the two proofs differ:

> "*For all integers* n*, if* $3n + 2$ *is even then* n *is even.*"

- **Proof by contraposition:** We show $odd(n) \Rightarrow odd(3n + 2)$.

  $odd(n) \Rightarrow n = 2k + 1$
  $\Rightarrow 3n + 2 = 3(2k + 1) + 2 = 6k + 3 + 2 = 6k + 4 + 1 = 2(3k + 2) + 1$
  $\Rightarrow odd(3n + 2)$.

- **Proof by contradiction:** Let $3n + 2$ be even and assume that n –contrary to the claim– is odd.

  Since $3n + 2$ is even, so is $3n$. We note that if we subtract an odd number from an even number, the result is odd. Since n is odd, $3n - n$ is odd. But $3n - n = 2n$ which is even. We have thus shown that our counter-assumption that n is odd leads to a contradiction.

8. For each of the following, determine whether it is valid or invalid. If valid then give a proof. If invalid then give a counter example.

   (a) $(A \cap B) \cup (C \cap D) \;=\; (A \cap D) \cup (C \cap B)$

   Solution: **Invalid**

   Counterexample: $A = \{a, b\},\ B = \{b, c\},\ C = \{d, e\},\ D = \{e, f\}$

   $(A \cap B) = \{b\},\ (C \cap D) = \{e\} \;\Rightarrow\; (A \cap B) \cup (C \cap D) \;=\; \{b, e\}$

   $(A \cap D) = \emptyset,\ (C \cap B) = \emptyset \;\Rightarrow\; (A \cap D) \cup (C \cap B) = \emptyset$

   (b) $A - (B \cup C) = (A - B) \cap (A - C)$

   Solution: **Valid** Algebraic proof:

   $$A - (B \cup C) \;=\; A \cap \overline{(B \cup C)} \;=\; A \cap (\overline{B} \cap \overline{C}) \;=$$
   $$(A \cap A) \cap (\overline{B} \cap \overline{C}) \;=\; A \cap (A \cap (\overline{B} \cap \overline{C})) \;=\; A \cap (A \cap \overline{B}) \cap \overline{C} \;=$$
   $$(A \cap \overline{B}) \cap (A \cap \overline{C}) \;=\; (A - B) \cap (A - C).$$

   (c) $B \cap C \subseteq A \;\;\Rightarrow\;\; (C - A) \cap (B - A)$ is empty

   Solution: **Valid** Element proof (by contradiction):

   Suppose $B \cap C \subseteq A$ and $(C - A) \cap (B - A) \neq \emptyset$.
   Then $\exists x : x \in (C - A)$ and $x \in (B - A) \;\;\Rightarrow$
   $x \in C$ and $x \notin A$ and $x \in B \;\;\Rightarrow$
   $x \in B \cap C$ and $x \notin A \;\;\Rightarrow$ contradiction, since $B \cap C \subseteq A$.

   (d) $(A \cup B) - (A \cap B) = A \;\;\Rightarrow\;\; B$ is empty

   Solution: **Valid** Element proof by contradiction (type 2):
   Suppose that $\big((A \cup B) \setminus (A \cap B)\big) = A$ and that $B \neq \emptyset$.
   Since $B \neq \emptyset$, let $x \in B$.
   Case 1: $x \in A$.
   Then, *a fortiori* $x \in A \cup B$, and since $x \in B$ (by our counter-assumption), we also have that $x \in A \cap B$. Thus $x \notin \big((A \cup B) \setminus (A \cap B)\big) = A$, meaning $x \notin A$; a contradiction of Case 1.

   Case 2: $x \notin A$.
   Then *a fortiori* $x \notin (A \cap B)$, and since $x \in B$ (by our counter-assumption), we also have that $x \in A \cup B$. Thus $x \in \big((A \cup B) \setminus (A \cap B)\big) = A$, meaning $x \in A$; a contradiction of Case 2.