CRAMER'S RULE

Let A be an invertible $n \times n$ matrix, $x = [x_1, x_2, ..., x_n]^T$. For any b in R^n, $Ax = b$ has a unique solution, and the entries of x are given by

$$x_i = \frac{\det A_i(b)}{\det A}, \ i = 1, 2, ..., n,$$

where $A_i(b)$ is the matrix obtained from A by replacing column i by the vector b.

Example: Let $A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Solve $Ax = b$.

Solution: $A_1(b) = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $A_2(b) = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$.

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix}} = \frac{3+10}{7+6} = 1,$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix}}{13} = \frac{35-9}{13} = \frac{26}{13} = 2.$$

The solution of $Ax = b$ is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Check: $$\begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Example: Use Cramer's rule to solve

$$\begin{aligned} x_1 + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8. \end{aligned}$$

Soultion: $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$,

$A_1(b) = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$, $A_2(b) = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}$,

$A_3(b) = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$.

$$\det A = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} \quad C_3' = C_3 - 2C_1$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -3 & 4 & 12 \\ -1 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 12 \\ -2 & 5 \end{vmatrix}$$

$$= 20 + 24 = 44$$

$$
\begin{aligned}
\det A_1(b) &= \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} \quad C_1' = C_1 - 3C_3 \\
&= \begin{vmatrix} 0 & 0 & 2 \\ 12 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} \\
&= (-1)^{1+3}\, 2 \begin{vmatrix} 12 & 4 \\ -1 & -2 \end{vmatrix} \\
&= 2(-24+4) = -40.
\end{aligned}
$$

Thus

$$
x_1 = \frac{\det A_1(b)}{\det A} = \frac{-40}{44} = \frac{-10}{11}.
$$

$$
x_2 = \frac{\det A_2(b)}{\det A} = \frac{72}{44} = \frac{18}{11}.
$$

$$
x_3 = \frac{\det A_3(b)}{\det A} = \frac{152}{44} = \frac{38}{11}.
$$

Example: Use Cramer's rule to solve for x without solving for y, z, and w.

$$\begin{aligned} -y+z+3w&=1\\ x+2y-z+w&=2\\ 3z+3w&=0\\ y+8z\qquad&=1 \end{aligned}$$

Solution: Corresponding matrix equation

$AX=b$ is

$$\begin{bmatrix} 0 & -1 & 1 & 3\\ 1 & 2 & -1 & 1\\ 0 & 0 & 3 & 3\\ 0 & 1 & 8 & 0 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix}=\begin{bmatrix} 1\\ 2\\ 0\\ 1 \end{bmatrix}.$$

$$A_1(b)=\begin{bmatrix} 1 & -1 & 1 & 3\\ 2 & 2 & -1 & 1\\ 0 & 0 & 3 & 3\\ 1 & 1 & 8 & 0 \end{bmatrix}.$$

$$
\begin{aligned}
\det A &= \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \\
&= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \quad R_3' = R_3 + R_1 \\
&= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \\
&= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} \\
&= 9 - 27 = -18.
\end{aligned}
$$

$$
\begin{aligned}
\det A_1(b) &= \begin{vmatrix} 1 & -1 & 1 & 3 \\ 2 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 8 & 0 \end{vmatrix} \quad C_3' = C_3 - C_4 \\
&= \begin{vmatrix} 1 & -1 & -2 & 3 \\ 2 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 8 & 0 \end{vmatrix} \\
= (-1)^{3+4}\ 3 & \begin{vmatrix} 1 & -1 & -2 \\ 2 & 2 & -2 \\ 1 & 1 & 8 \end{vmatrix} \quad \begin{matrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{matrix} \\
&= -3 \begin{vmatrix} 1 & -1 & -2 \\ 0 & 4 & 2 \\ 0 & 2 & 10 \end{vmatrix} \\
&= -3 \begin{vmatrix} 4 & 2 \\ 2 & 10 \end{vmatrix} \\
&= -3(40 - 4) = -108.
\end{aligned}
$$

So, $x = \dfrac{\det A_1(b)}{\det A} = \dfrac{-108}{-18} = 6.$

Similarly,

$$y = \frac{\det A_2(b)}{\det A} = \frac{30}{-18} = \frac{-5}{3},$$

$$z = \frac{\det A_3(b)}{\det A} = \frac{-6}{-18} = \frac{1}{3},$$

$$w = \frac{\det A_4(b)}{\det A} = \frac{6}{-18} = \frac{-1}{3}.$$

A FORMULA FOR A^{-1}

Let A be an invertible $n \times n$ matrix.
Let A_{ij} be the submatrix of A formed by deleting row i and column j.

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

is called the $(i, j) - cofactor$ of A.

Cofactor expansion about the i th row:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Cofactor expansion about the j th column:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

The transpose of the matrix of cofactors from A is called the adjoint of A, i.e,

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

$$\boxed{A^{-1} = \frac{1}{\det A} \text{adj}A}$$

$$\boxed{\text{adj}A = \det A \; A^{-1}}$$

Homework: Show that $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$.

Example: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$.

Find adj$A$ and A^{-1}.

Solution:

$$\begin{aligned}
C_{11} &= (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12, \\
C_{12} &= (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6, \\
C_{13} &= (-1)^{1+3} \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -16, \\
C_{21} &= (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = 4, \\
C_{22} &= (-1)^{2+2} \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 2, \\
C_{23} &= (-1)^{2+3} \begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = 16,
\end{aligned}$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} = 12,$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -10,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = 16.$$

$$\begin{aligned}
\text{adj}A &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T \\
&= \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}^T \\
&= \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.
\end{aligned}$$

$$\begin{aligned}
\det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= 3 \cdot 12 + 2 \cdot 6 + (-1) \cdot (-16) \\
&= 36 + 12 + 16 \\
&= 64.
\end{aligned}$$

Thus,

$$\begin{aligned}
A^{-1} &= \frac{1}{\det A}\,\mathrm{adj}A \\
&= \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} \\
&= \begin{bmatrix} 12/64 & 4/64 & 12/64 \\ 6/64 & 2/64 & -10/64 \\ -16/64 & 16/64 & 16/64 \end{bmatrix} \\
&= \begin{bmatrix} 3/16 & 1/16 & 3/16 \\ 3/32 & 1/32 & -5/32 \\ -1/4 & 1/4 & 1/4 \end{bmatrix}.
\end{aligned}$$

Homework: Find the adjoint of A, and then the inverse of A.

i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **ii)** $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 9 \\ -1 & 3 & 0 \end{bmatrix}$

Ans: $A^{-1} = \begin{bmatrix} 3/5 & 0 & -2/5 \\ 1/5 & 0 & 1/5 \\ -11/5 & 1/9 & 1/5 \end{bmatrix}$