

MAT 1348A — Third Homework Assignment — Due Mar. 5, 2014 at 11:20am

Instructions: Show all relevant work to receive full credit. Submit a finished product, not a draft. You may write on both sides of the paper or insert additional pages if necessary. Please staple the pages. Submit the assignment in the appropriate submission box in the Department of Mathematics and Statistics. Late assignments will not be accepted.

1. Use a **proof by cases** to show the following:

Let n be an integer. If 3 does not divide n , then 3 divides $n^3 - 2n^2 + 5n - 10$.

Let p : "3 does not divide n "

q : "3 divides $n^3 - 2n^2 + 5n - 10$."

Furthermore, let p_1 : " $n = 3k + 1$ for some integer k "

p_2 : " $n = 3k + 2$ for some integer k ."

We must prove $p \rightarrow q$, where $p = p_1 \vee p_2$.

Since $(p_1 \vee p_2 \rightarrow q) \equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q)$, as shown in class, we must prove both $p_1 \rightarrow q$ and $p_2 \rightarrow q$.

To prove $p_1 \rightarrow q$:

Assume $n = 3k + 1$ for $k \in \mathbb{Z}$. Then:

$$\begin{aligned} n^3 - 2n^2 + 5n - 10 &= (3k+1)^3 - 2(3k+1)^2 + 5(3k+1) - 10 \\ &= 27k^3 + 27k^2 + 9k + 1 - 2(9k^2 + 6k + 1) \\ &\quad + 5(3k+1) - 10 \\ &= 27k^3 + 9k^2 + 12k - 6 \\ &= 3(9k^2 + 3k^2 + 4k - 2) \end{aligned}$$

Since $9k^2 + 3k^2 + 4k - 2$ is an integer, we conclude that $3 \mid (n^3 - 2n^2 + 5n - 10)$.

To prove $p_2 \rightarrow q$:

Assume $n = 3k+2$ for $k \in \mathbb{Z}$. Then:

$$\begin{aligned}n^3 - 2n^2 + 5n - 10 &= (3k+2)^3 - 2(3k+2)^2 + 5(3k+2) - 10 \\&= 27k^3 + 54k^2 + 36k + 8 \\&\quad - 2(9k^2 + 12k + 4) + 5(3k+2) - 10 \\&= 27k^3 + 36k^2 + 27k = 3(9k^3 + 12k^2 + 9k)\end{aligned}$$

Since $9k^3 + 12k^2 + 9k$ is an integer, we conclude that $3 \mid (n^3 - 2n^2 + 5n - 10)$.

Conclusion: since $p_1 \rightarrow q$ and $p_2 \rightarrow q$, we have that $(p_1 \vee p_2) \rightarrow q$ as required. \square

2. Let A, B, C be three subsets of the universal set.

(a) Prove the following using set identities and properties of set operations:

$$(A - B) \cup (A - \bar{C}) = A - \overline{B \cup C}.$$

$$\begin{aligned}(A - B) \cup (A - \bar{C}) &= (A \cap \bar{B}) \cup (A \cap \bar{\bar{C}}) \\ &= (A \cap \bar{B}) \cup (A \cap C) \\ &= A \cap (\bar{B} \cup C) \\ &= A \cap \overline{B \cap \bar{C}} \\ &= A - (B \cap \bar{C}) \\ &= A - \overline{\bar{B} \cup C}\end{aligned}$$

(b) For each of the following, either prove the statement or give a counterexample. (A counterexample for (i) would consist of particular sets A, B, C such that $A - B = A - C$ but $B \neq C$.)

- i. If $A - B = A - C$, then $B = C$.
- ii. If $A - B = A - C$, then $A \cap B = A \cap C$.

(i) False. Counterexample:

$$A = \{1, 2\}$$

$$B = \{2, 3\}$$

$$C = \{2, 4\}$$

Then $A - B = \{1\} = A - C$, but $B \neq C$.

(ii) True. Proof:

We shall assume $A - B = A - C$, and prove $A \cap B = A \cap C$. That is, we must prove that $A \cap B \subseteq A \cap C$ and $A \cap C \subseteq A \cap B$. However, since the statement to prove is symmetric in B and C , it suffices to prove $A \cap B \subseteq A \cap C$.

Take any $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

If $x \notin C$, then $x \in A - C$, and hence $x \in A - B$,
and hence $x \notin B$, a contradiction.

Thus $x \in C$, and so $x \in A \cap C$.

It follows that $A \cap B \subseteq A \cap C$.

Again, by symmetry, we also have $A \cap C \subseteq A \cap B$.

We conclude that $A \cap B = A \cap C$. □

3. (a) Is the function

$$f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}, \quad f(x, y) = 2x - y$$

one-to-one? Fully justify your answer.

Not one-to-one. Counterexample:

$$f(1, 2) = 0 = f(2, 4).$$

(b) Is the function

$$g: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}, \quad g(x, y) = (3x + y, x)$$

invertible? If so, find its inverse. Fully justify your answer.

g is invertible $\Leftrightarrow g$ is a bijection
 $\Leftrightarrow (g \text{ is one-to-one and onto}).$

To show g is one-to-one:

Take any $(x_1, y_1), (x_2, y_2) \in \mathbb{Z} \times \mathbb{Z}$.

Suppose $g(x_1, y_1) = g(x_2, y_2)$

Then $(3x_1 + y_1, x_1) = (3x_2 + y_2, x_2)$,

$$\text{so } \begin{cases} 3x_1 + y_1 = 3x_2 + y_2 \\ x_1 = x_2 \end{cases}$$

It follows that $x_1 = x_2$ and $y_1 = y_2$, so $(x_1, y_1) = (x_2, y_2)$.

We conclude g is one-to-one.

To show g is onto:

Take any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ (codomain). We must find $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ (domain) s.t. $g(x, y) = (a, b)$.

That is: $(3x+y, x) = (a, b)$

$$\text{or } \begin{cases} 3x+y = a \\ x = b \end{cases}$$

The solution is $x=b, y=a-3b$.

Since $a, b \in \mathbb{Z}$, x and y are integers.

Thus $(x, y) = (b, a-3b) \in \mathbb{Z} \times \mathbb{Z}$ (domain)

Since for any (a, b) in the codomain we can find (x, y) in the domain s.t. $g(x, y) = (a, b)$, we conclude that g is onto.

Thus g is invertible.

Inverse: $g^{-1}: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$

$$g^{-1}(a, b) = (b, a-3b).$$

4. Let \mathcal{R} be a binary relation on the set $\{1, 2, 3, 4\}$ defined by

$$\mathcal{R} = \{(1, 1), (1, 4), (2, 1), (2, 2), (2, 4), (3, 1), (3, 2), (3, 3)\}$$

Determine whether \mathcal{R} is reflexive, symmetric, antisymmetric, or transitive. *Fully justify your answer.*

* Not reflexive because $(4, 4) \notin \mathcal{R}$.

* Not symmetric because e.g. $(1, 4) \in \mathcal{R}$ but $(4, 1) \notin \mathcal{R}$.

* Antisymmetric: for any $a, b \in \{1, 2, 3, 4\}$,
 $((a, b) \in \mathcal{R} \text{ and } (b, a) \in \mathcal{R}) \rightarrow a = b$

is true because

$$(1, 4) \in \mathcal{R} \text{ but } (4, 1) \notin \mathcal{R}$$

$$(2, 1) \in \mathcal{R} \text{ but } (1, 2) \notin \mathcal{R}$$

$$(2, 4) \in \mathcal{R} \text{ but } (4, 2) \notin \mathcal{R}$$

$$(3, 1) \in \mathcal{R} \text{ but } (1, 3) \notin \mathcal{R}$$

$$(3, 2) \in \mathcal{R} \text{ but } (2, 3) \notin \mathcal{R}$$

* Not transitive because e.g.

$$(3, 2) \in \mathcal{R} \text{ and } (2, 4) \in \mathcal{R} \text{ but } (3, 4) \notin \mathcal{R}.$$



5. A relation \mathcal{R} is defined on the set \mathbb{R}^2 as follows:

$$(x_1, y_1) \mathcal{R} (x_2, y_2) \leftrightarrow 3x_1 - y_1 = 3x_2 - y_2.$$

(a) Prove that \mathcal{R} is an equivalence relation on \mathbb{R}^2 .

(b) Determine the partition of the set A below into equivalence classes of \mathcal{R} (restricted to A).

$$A = \left\{ (-2, -9), (-\sqrt{3}, 4), \left(-\frac{4}{3}, 3\sqrt{3}\right), (0, 0), (0, -2), (1, 0), (1, 1), (\sqrt{3}, -4), (2, 3), (2, 4) \right\}$$

(c) Give a geometric description of the equivalence class $[(1, 1)]_{\mathcal{R}}$.

(a) * reflexive: for all $(x, y) \in \mathbb{R}^2$:

$$3x - y = 3x - y, \text{ so } (x, y) \mathcal{R} (x, y)$$

* symmetric: for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$:

$$(x_1, y_1) \mathcal{R} (x_2, y_2) \rightarrow 3x_1 - y_1 = 3x_2 - y_2$$

$$\rightarrow 3x_2 - y_2 = 3x_1 - y_1$$

$$\rightarrow (x_2, y_2) \mathcal{R} (x_1, y_1)$$

* transitive: for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$:

$$\left((x_1, y_1) \mathcal{R} (x_2, y_2) \text{ and } (x_2, y_2) \mathcal{R} (x_3, y_3) \right)$$

$$\rightarrow (3x_1 - y_1 = 3x_2 - y_2 \text{ and } 3x_2 - y_2 = 3x_3 - y_3)$$

$$\rightarrow 3x_1 - y_1 = 3x_3 - y_3 \rightarrow (x_1, y_1) \mathcal{R} (x_3, y_3)$$

since \mathcal{R} is reflexive, symmetric, and transitive,
it is an equivalence relation.

(b) Equivalence classes in this partition:

$$[(-2, -9)]_{\mathcal{R}} = \left\{ (-2, -9), (1, 0), (2, 3) \right\}$$

$$[(-\sqrt{3}, 4)]_{\mathbb{R}} = \left\{ (-\sqrt{3}, 4), \left(-\frac{4}{3}, 3\sqrt{3}\right) \right\}$$

$$[(0, 0)]_{\mathbb{R}} = \{ (0, 0) \}$$

$$[(0, -2)]_{\mathbb{R}} = \{ (0, -2), (1, 1), (2, 4) \}$$

$$[(\sqrt{3}, -4)]_{\mathbb{R}} = \{ (\sqrt{3}, -4) \}$$

Note that

$$[(a, b)] = \left\{ (x, y) \in \mathbb{R}^2 : 3x - y = 3a - b \right\}$$

$$(c) [(1, 1)]_{\mathbb{R}} = \{ (x, y) \in \mathbb{R}^2 : 3x - y = 2 \}$$

$$= \{ (x, y) \in \mathbb{R}^2 : y = 3x - 2 \}$$

= straight line with equation $y = 3x - 2$