② It is possible to show that although the SL problem is singular, the basis of functions

$$\{ J_m(z_{mn}x) \}_{n=1\cdots\infty}$$

still forms a complete orthogonal basis for all non-singular functions in $(0,1)$ vanishing at $x=1$

$\Rightarrow$ we can always write

$$a_m(x) = \sum_{n=1}^{\infty}{}' \alpha_{mn} J_m(z_{mn}x)$$

$$b_m(x) = \sum_{n=1}^{\infty}{}' \beta_{mn} J_m(z_{mn}x)$$

with

$$\alpha_{mn} = \int_0^1 x a_m(x) J_m(z_{mn}x) \, dx \cdot \frac{1}{\langle J_m(z_{mn}x), J_m(z_{mn}x) \rangle}$$

$$\beta_{mn} = \int_0^1 x b_m(x) J_m(z_{mn}x) \, dx \cdot \frac{1}{\langle J_m(z_{mn}x), J_m(z_{mn}x) \rangle}$$

and $\langle J_m(z_{mn}x), J_m(z_{mn}x) \rangle = \int_0^1 x J_m^2(z_{mn}x) \, dx$

$\Rightarrow$ once this is done, the coefficients A_{mn} and B_{mn} can be identified with this and the solution to the problem is

$$u(x,\theta,t) = \sum_{n,m} \left(\frac{\alpha_{mn}}{z_{mn}c} \cos m\theta + \frac{\beta_{mn}}{z_{mn}c} \sin m\theta \right) J_m(z_{mn}x) \sin(z_{mn}ct).$$

Simple Example: Suppose $F(x,\theta) = 4 J_2 \left(z_{32} x \right) \cos 2\theta$

$\Rightarrow$ here F is already in the correct form with

$\beta_{mn} = 0 \quad \forall m,n$

$\alpha_{23} = 4$ and $\alpha_{mn} = 0$ for all others m,n

$$\Rightarrow u(x,\theta,t) = \frac{4}{z_{32}c} \cos 2\theta \, J_2(z_{32}x) \sin(z_{32}ct).$$

6.7 Non-homogeneous equations; introduction to Green's functions

6.7.1 Non homogeneous (regular) SL problems (ODEs)

Given the ODE $\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x)$

with bcs $\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(b) + \delta u'(b) = 0 \end{cases}$

① Seek solutions of the homogeneous eigenvalue eq.

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = -\lambda u$$

→ this yields the eigenfunctions $\{v_n\}$ and eigenvalues $\{\lambda_n\}$

② Write $F(x) = \sum_n b_n v_n(x)$

(with $b_n = \int_a^b F(x')v_n(x')dx'$, if the v_ns are properly normalized)

Then since we know that the solution can also be written as

$$u(x) = \sum_n a_n v_n(x)$$

we can write

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = \sum_n -\lambda_n a_n v_n(x) = \sum_n b_n v_n(x)$$

and by identification, $a_n = -\frac{b_n}{\lambda_n}$

$$\Rightarrow \quad u(x) = \sum_n -\frac{b_n}{\lambda_n} v_n(x) = -\sum_n \int_a^b \frac{1}{\lambda_n} F(x')v_n(x')v_n(x)dx'$$

$$= -\int_a^b G(x;x')F(x')dx'$$

where $\quad G(x;x') = \sum_n \frac{1}{\lambda_n} v_n(x') v_n(x)$

- $G(x;x')$ is called the Green's function of the S.L problem

- It only depends on the characteristics of the homogeneous problem ($\{v_n\}$, $\{\lambda_n\}$) but, when integrated through with the forcing term $F(x)$, yields the solution of the forced problem

- Note that if the $\{v_n\}$ are not normalized then

$$G(x;x') = \sum_n \frac{1}{\|v_n\|^2} \frac{1}{\lambda_n} v_n(x') v_n(x) .$$

$$\text{where } \|v_n\|^2 = \int_a^b r(x) v_n(x)^2 \, dx .$$

<u>Example</u> Consider

$$y'' + y = 3\sin(2\pi x) \qquad \begin{array}{l} y(0) = 0 \\ y(1) = 0 \end{array}$$

We seek the eigenfunctions of $y'' + y = -\lambda y$

$$\rightarrow \quad y'' + (1+\lambda) y = 0$$

$$\text{so} \quad y = \alpha \cos(\sqrt{1+\lambda}\, x) + \beta \sin(\sqrt{1+\lambda}\, x)$$

$$\text{with } \begin{cases} \alpha = 0 \\ \sqrt{1+\lambda_n} = n\pi \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_n = n^2\pi^2 - 1 \\ v_n(x) = \sin(n\pi x) \end{cases}$$

$\Rightarrow$ The Green's function $\displaystyle G(x,x') = \sum_n \frac{\sin(n\pi x)\sin(n\pi x')}{\lambda_n \|\sin n\pi x\|}$

$$= \sum_n \frac{2}{n^2\pi^2 - 1} \sin(n\pi x)\sin(n\pi x')$$

so the solution to the problem is $y(x) = -\int_0^1 G(x,x')F(x')dx'$

$$y(x) = \int_0^1 \sum_n \frac{3\sin(2\pi x')}{1-n^2\pi^2} 2\sin(n\pi x)\sin(n\pi x')\,dx'$$

$$= \frac{3}{1-4\pi^2}\sin(2\pi x)$$

6.7.2 Application to parabolic/hyperbolic PDEs.

Now consider either $u_t - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$

or $u_{tt} - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$.

Idea: Solve the associated Sturm-Liouville problem

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] + \lambda u = 0$$

to find the eigenvalues and eigenfunctions $\{v_n\}$, $\{\lambda_n\}$

then expand

$$F(x,t) = \sum_n b_n(t)v_n(x)$$

(in this case, $b_n(t) = \int_a^b F(x,t)r(x)v_n(x)\,dx$)

Assume a solution of the form

$$u(x,t) = \sum_n a_n(t)v_n(x)$$

and try the ansatz into the equation:

(example: parabolic) $\displaystyle\sum_n \dot{a}_n(t)v_n(x) - \frac{1}{r(x)}\left[\left(p(x)\sum_n a_n(t)v_n'(x)\right)' + q(x)\sum a_n(t)v_n(x)\right]$

$$= \sum_n b_n(t)v_n(x)$$

so that

$$\sum_n{}' \dot{a}_n(t)v_n(x) + \sum_n{}' \lambda_n a_n(t)v_n(x) = \sum_n b_n(t)v_n(x)$$

and (by orthogonality):

$$\dot{a}_n + \lambda_n a_n = b_n(t)$$

↳ integrating factor method:

$$\frac{d}{dt}\left(a_n e^{\lambda_n t}\right) = b_n(t)\, e^{\lambda_n t}$$

so $$a_n(t)e^{\lambda_n t} - a_n(0) = \int_0^t b_n(t')e^{\lambda_n t'}\,dt'$$

$$\Rightarrow a_n(t) = a_n(0)e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t b_n(t')e^{\lambda_n t'}\,dt'.$$

Putting it all together we find that

$$u(x,t) = \sum_n{}' a_n(0)e^{-\lambda_n t}v_n(x) + \int_0^t \sum_n{}' e^{-\lambda_n(t-t')}v_n(x)\, b_n(t')\,dt'$$

$$= \sum_n{}' a_n(0)e^{-\lambda_n t}v_n(x) + \int_0^t\int_a^b \sum_n{}' e^{-\lambda_n(t-t')}v_n(x)v_n(x')r(x')\,F(x',t')\,dx'\,dt'$$

So we can write

$$u(x,t) = \sum_n a_n(0)e^{-\lambda_n t}v_n(x) + \int_0^t\int_a^b G(x,t;x',t')F(x',t')\,dx'\,dt'$$

with $$G(x,t;x',t') = \sum_n{}' e^{-\lambda_n(t-t')}v_n(x)v_n(x')r(x')$$

↳ another example of a Green's function.

↳ u is the sum of
- the solution to the problem with no forcing
- \+ the weighted integral of $F(x,t)$ with the Green's function.