6.6 Example of application: wave in a non-homogeneous medium

Consider the wave equation for varying wave speed:

$$\frac{\partial^2 u}{\partial t^2} = c^2(x) \frac{\partial^2 u}{\partial x^2} \qquad c^2(x) > 0 \ \forall x \in [0,1].$$

on a finite string : $x \in [0,1]$
with $u(0) = u(1) = 0 \ \forall t$

Then this is an archetype S.L problem / eigenfunction expansion problem.

let $u = T(t) F(x)$ then

$$\frac{\ddot{T}}{T} = -\lambda \qquad c^2(x) \frac{F''}{F} = -\lambda$$

so $$F'' = -\frac{\lambda F}{c^2(x)},$$ a Sturm-Liouville problem with

$$\begin{cases} p(x) = 1 \\ q(x) = 0 \\ r(x) = \dfrac{1}{c^2(x)} \end{cases}$$

What can we learn from this?

① $\lambda \geq 0$.

Indeed: $$R(u) = \frac{\int_0^1 u'^2 \, dx}{\int_0^1 \frac{u^2}{c^2(x)} \, dx} \geq 0$$ for any function u.

② Some estimate of the fundamental mode of vibration can be made by minimizing $R(u)$: $f_0 = \sqrt{\lambda_0}$ with

$$0 \leq \lambda_0 \leq \frac{\int_0^1 \bar{u}'^2 \, dx}{\int_0^1 \frac{\bar{u}^2}{c^2(x)} \, dx}$$

(i.e. try $\bar{u} = \sin(\pi x)$ or $\bar{u} = x(1-x)$)

③ Some estimate of the high frequencies of vibration can be made: $f_n \simeq \sqrt{\lambda_n}$ with

$$\lambda_n \simeq \left(\frac{n\pi}{\int_0^1 \frac{1}{c(x)}\, dx} \right)^2$$

<u>Example</u>: suppose we consider a string with a slight defect at $x = x_0$

$$c(x) = c_0\left(1 + \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right)$$

then

$$\frac{1}{c(x)} \simeq \frac{1}{c_0}\left(1 - \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right)$$

and for large n

$$\lambda_n \simeq n^2\pi^2 \left[\int_0^1 \frac{1}{c_0}\left(1 - \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right) dx \right]^{-2}$$

- if $\sigma \ll 1$ then the width of the Gaussian is small enough that we can approximate

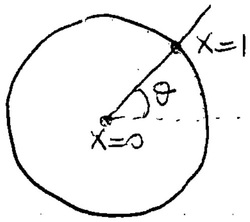
$$\int_0^1 e^{-\frac{(x-x_0)^2}{2\sigma^2}}\, dx \simeq \int_{-\infty}^{+\infty} e^{-\frac{(x-x_0)^2}{2\sigma^2}}\, dx \simeq \sqrt{2\pi}\,\sigma$$

In that case
$$\lambda_n \simeq n^2\pi^2 c_0^2 \left(1 - \sqrt{2\pi}\,\sigma\epsilon\right)^{-2}$$
$$\simeq n^2\pi^2 c_0^2 \left(1 + 2\sqrt{2\pi}\,\sigma\epsilon\right)$$

→ by comparing the frequencies "observed" on the imperfect string to those from a theoretical "perfect" string, we can deduce $\underline{\underline{\sigma\epsilon}}$ but not x_0.

6.7 Example of standard S.L problems: Bessel Equations

Given a circular drum, displacements of the membrane satisfy a wave equation of the kind



$$u_{tt} = c^2 \nabla^2 u = c^2 \left[\frac{1}{x} \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x} \right) + \frac{1}{x^2} \frac{\partial^2 u}{\partial \theta^2} \right]$$

with initial conditions

$$u_t(x,\theta,0) = F(x,\theta)$$
$$u(x,\theta,0) = 0$$

and boundary cs $\quad u(1,\theta,t) = 0$

We consider again solutions of the kind

$$u(x,\theta,t) = R(x)\,\Theta(\theta)\,T(t)$$

$$\Rightarrow \quad \frac{\ddot{T}}{T} = c^2 \left[\frac{1}{R} \frac{1}{x} \frac{d}{dx}\left(x \frac{dR}{dx} \right) + \frac{1}{x^2} \frac{1}{\Theta} \frac{d^2\Theta}{d\theta^2} \right]$$

a function of time only

a function of x and θ only

$$\Rightarrow \quad \begin{cases} \ddot{T} = -c^2 k^2 T \\ \dfrac{1}{R} \dfrac{1}{x} \dfrac{d}{dx}\left(x \dfrac{dR}{dx} \right) + \dfrac{1}{x^2} \dfrac{1}{\Theta} \dfrac{d^2\Theta}{d\theta^2} = -k^2 \end{cases}$$

(expect oscillations)

$$\Rightarrow \quad \frac{1}{R}\, x \frac{d}{dx}\left(x \frac{dR}{dx} \right) + \frac{1}{\Theta} \frac{d^2\Theta}{d\theta^2} = -k^2 x^2$$

$$\Rightarrow \quad \frac{1}{R}\, x \frac{d}{dx}\left(x \frac{dR}{dx} \right) + k^2 x^2 = -\frac{1}{\Theta} \frac{d^2\Theta}{d\theta^2}$$

a function of x only

a function of θ only

$$\Rightarrow \quad \begin{cases} \dfrac{1}{\Theta} \dfrac{d^2\Theta}{d\theta^2} = -m^2 \\ x \dfrac{d}{dx}\left(x \dfrac{dR}{dx} \right) + k^2 x^2 R = m^2 R \end{cases}$$

(expect solutions to be $\cos m\theta$, $\sin m\theta$)

$\Rightarrow$ Since the drum is circular, the θ- solutions must be periodic in θ so

$$\Theta_m = a_m \cos m\theta + b_m \sin m\theta$$

So the R-equation is a S.L problem of the kind

$$\frac{d}{dx}\left(x\frac{dR}{dx}\right) - \frac{m^2}{x}R = -k^2 x R$$

$$\Rightarrow \begin{cases} p(x) = x \\ q(x) = -\frac{m^2}{x} \\ r(x) = x \end{cases} \quad \text{with eigenvalue } \lambda = k^2.$$

in the interval $[0, 1]$

<u>Note</u> : This SL problem is singular ; we expect properties $1 \to 3$ to be true (i.e. e.values are real and e-functions of $\neq$ e-values are orthogonal wrt the inner product

$$\langle u, v \rangle = \int_0^1 r(x) u(x) v(x) dx$$

However, the e-values need not be simple.

To find out more about the problem note that for each m

$$(*) \qquad \frac{d}{dx}\left(x\frac{dR_m}{dx}\right) - \frac{m^2}{x}R_m = -k^2 x R_m$$ is in fact related

to a Bessel equation, with well-know & studied solutions

Inded Let $\xi = kx$ then $(*)$ becomes

$$\xi^2 \frac{d^2 R_m}{d\xi^2} + \xi \frac{dR_m}{d\xi} + (\xi^2 - m^2) R_m = 0$$

$\uparrow$ <u>A Bessel equation of degree m</u>

Solutions to the Bessel equation of degree m are the <u>Bessel functions</u>. General solutions to the Bessel equation are

$$R_m = \alpha J_m(\xi) + \beta Y_m(\xi) = \alpha J_m(kx) + \beta Y_m(kx)$$

the regular Bessel function of degree m or Bessel function of the first kind

the singular Bessel function of degree m. or Bessel function of the second kind

<u>Applying Boundary conditions</u>

- we want $R_m(x)$ to be regular at the origin and $R_m = 0$ for $x = 1$

① The function $Y_m(x)$ is singular at the origin ($Y_m(x) \to -\infty$ as $x \to 0$) so $\beta = 0$

② $R_m(x) = \alpha J_m(kx)$

$R_m(1) = 0 \iff \alpha J_m(kx) = 0$

this implies that k must be a zero of the Bessel function J_m.

looking at the diagrams for J_m we see that $J_m(x)$ has an ∞ of zeros. Let's call them z_{mn} $n = 1 \dots \infty$

such that $J_m(z_{mn}) = 0$

$\Rightarrow k_n = z_{mn}$ the z_{mn} are well know numbers found in all good Textbooks.

So finally, $R_m(x)$ is in fact a linear combination of $J_m(z_{mn}x)$ functions.

The temporal solution, for each k_n, and each m

$$a_{mn} \cos(k_{mn} ct) + b_{mn} \sin(k_{mn} ct)$$
$$= a_{mn} \cos(z_{mn} ct) + b_{mn} \sin(z_{mn} ct)$$

Initial conditions imply that $a_{mn} = 0$ so finally,

$$u(x,\theta,t) = \sum_{n,m}{}' \left(A_{mn} \cos m\theta + B_{mn} \sin m\theta \right) J_m(k_n x) \sin(z_{mn} ct)$$

To fit the last initial condition $u_t(x,\theta,0) = F(x,\theta)$ we see that

$$u_t(x,\theta,0) = \sum_{n,m}{}' \left(A_{mn} \cos m\theta + B_{mn} \sin m\theta \right) J_m(\cancel{mx}) k_n c$$
$$= F(x,\theta).$$

$\Rightarrow$ provided we can decompose the initial condition function $F(x,\theta)$ on the basis of $\{\cos m\theta, \sin m\theta\}$ and $\{J_m(k_n x)\}$ we can find the coefficients A_{mn} and B_{mn}.

How to do it? : ① $F(x,\theta)$ is necessarily periodic in θ at all values of x.

So, we can always write, for each value of x

$$F(x,\theta) = \sum_{m}{}' a_m(x) \cos m\theta + b_m(x) \sin m\theta$$

where

$$a_m(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x,\theta) \cos m\theta \, d\theta \quad \text{if } m > 0$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x,\theta) \, d\theta \quad \text{if } m = 0$$
$$b_m(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x,\theta) \sin m\theta \, d\theta$$