Solutions

and Marking Scheme

(white)

MAT 1322 C W2010 Wednesday, Feb. 3rd 17:30–18:50
Instructor:Termeh Kousha

MIDTERM TEST 1

Max = 20

Student Number: ______________________________

•Time: 80 min.

• Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite.

• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

1.[4 points] (a). Consider the integral $\int_2^{\infty} 2e^{-2x}\,dx$. Does it converge or diverge? If it converges, give its value.

$$\int_2^{\infty} 2e^{-2x}\,dx = \lim_{t\to\infty} \int_2^{t} 2e^{-2x}\,dx$$ 0.5 points

$$= \lim_{t\to\infty}\left(-e^{-2x}\Big|_2^t\right) = \lim_{t\to\infty}\left(-e^{-2t} + e^{-4}\right)$$

0.5 point

$$= \lim_{t\to\infty}\left(-\overset{0}{e^{-2t}}\right) + e^{-4}$$

1 points

$$= e^{-4}$$ It converges to e^{-4}

(b) Use the Comparison Test to determine if the integral $\int_0^1 \frac{2+2\cos(x)}{2x^2+\sqrt{x}}\,dx$ converges or diverges. If it converges, give an upper bound for its value.

0.5 $\quad 2\cos x \leqslant 2 \rightarrow 2\cos x + 2 \leqslant 4$

As $x \to 0$ $\quad 2x^2+\sqrt{x}$ behaves like $\sqrt{x}$

For $0 \leqslant x \leqslant 1 \rightarrow \sqrt{x} + 2x^2 \geqslant \sqrt{x}$ so $\frac{1}{\sqrt{x}+2x^2} \leqslant \frac{1}{\sqrt{x}}$

0.5 $\rightarrow$ so $\int_0^1 \frac{2+2\cos(x)}{2x^2+\sqrt{x}}\,dx \leqslant \int_0^1 \frac{4}{\sqrt{x}}\,dx$

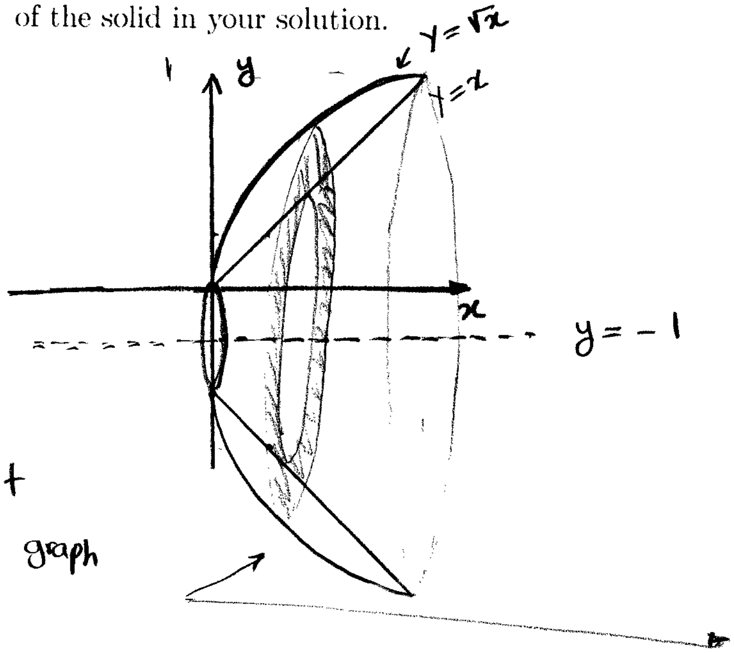
$$\int_0^1 \frac{4}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{4}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 4\left(2x^{1/2}\Big|_t^1\right)$$

1 point

$$= \lim_{t\to 0^+}\left(8 - 8\sqrt{t}\right) = 8 - \lim_{t\to 0^+}\left(8\sqrt{t}\right) = 8$$

It Converges with upperbound 8.

2.[4 points] Find the volume of the solid obtained by the curves $y = x$, $y = \sqrt{x}$, is rotated around the line $y = -1$. Include a sketch of the region and a typical cross-section of the solid in your solution.

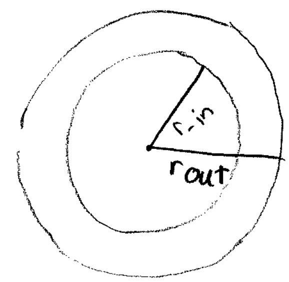


1 point for graph

Points of intersection:

$x = \sqrt{x} \Rightarrow x = 0, 1.$ 0.5 points.

Cross section area prependicular to x-axis is a washer



$r\_in = x+1$

$rout = \sqrt{x} + 1.$

$A(x) = \pi (r\_out)^2 - \pi (r\_in)^2 =$

$\pi(\sqrt{x}+1)^2 - \pi(x+1)^2 = \pi(x + 2\sqrt{x} + 1 - x^2 - 2x - 1)$

1points.

$= \pi(-x^2 - x + 2\sqrt{x})$

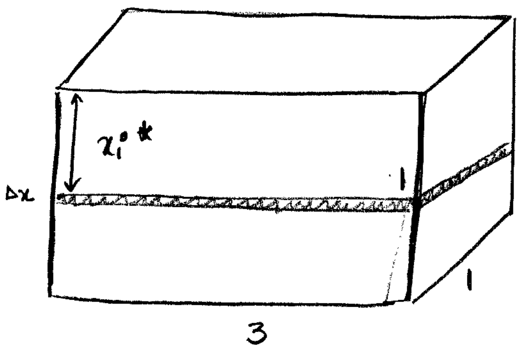
$V = \int_0^1 A(x)\,dx = \pi \int_0^1 (-x^2 - x + 2\sqrt{x})\,dx.$

1points

$= \pi\left(-\frac{x^3}{3} - \frac{x^2}{2} + \frac{4}{3}x^{3/2} \Big|_0^1\right) = \boxed{\frac{\pi}{2}}$

0.5 final answer

3.[4 points] An aquarium 3 m long, 1 m wide, and 1m deep is full of water. Find the work needed to pump two third of the water out of the aquarium. The density of water is $\rho = 1000\ \mathrm{kg/m^3}$ and the acceleration of gravity is $g = 9.8\ \mathrm{m/s^2}$. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.



n: the number of subintervals of length Δx

x_i^* . sample point $[x_{i-1}, x_i]$.

$A_i = 3 \times 1 = 3\ m^2$

volume $= 3\Delta x\ m^3$

$m_i = V_i\rho = 3\Delta x \times 1000 = 3000\ \Delta x$

$F_i = m_i g = 3000 \times 9.8\ \Delta x.$

$= 29400\ \Delta x\ N$

$W = \lim_{n \to \infty} \sum_{i=1}^{n} 29400\ \Delta x\ x_i^* =$

for $0 \leq x_i \leq 2/3$

$\int_0^{2/3} 29400\ x\, dx = 14700\, x^2 \Big|_0^{2/3}$

$= 14700 \cdot \frac{4}{9} \simeq 6533.3\ J$

4.[3 points] Give the integral for the arclength of the curve $x = 2y^{3/2} + 1$ between $y = 0$ and $y = 1$ and then evaluate it.

$$f'(y) = 2 \cdot \frac{3}{2} y^{1/2} \rightarrow f'(y)^2 = 9y$$ ← 1 point

$$L = \int_0^1 \sqrt{9y + 1}\, dy$$ ← 1 point

$$= \frac{2}{3} \times \frac{1}{9} (9y+1)^{3/2} \Big|_0^1 =$$

$$\frac{2}{27}\left(10^{3/2} - 1\right)$$

1point

5.[3 points] Give the integral for the average value of the function $f(x) = xe^x$ on the interval $[0, 3]$ and then evaluate it.

$$A(x) = \frac{1}{3-0}\int_0^3 xe^x\,dx \quad \leftarrow \text{1 point}$$

1 point $\longrightarrow$

$$= \frac{1}{3}\left[xe^x\Big|_0^3 - \int_0^3 e^x\,du \right]$$

integration by parts

$x = u \rightarrow du = 1$

$e^x du = dv$

$e^u = v$

$$= \frac{1}{3}\left[3e^3 - e^3 + 1 \right]$$

$$\frac{1}{3}\left[2e^3 + 1 \right]$$

1 point

6.[2 points] Solve the initial value problem: $\sqrt{1-x^2}\,\frac{dy}{dx} = 3y\,, \quad y(0) = 2\,.$

$$\frac{\sqrt{1-x^2}}{dx} = \frac{3y}{dy} \longrightarrow \frac{dy}{3y} = \frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{dy}{3y} = \int \frac{dx}{\sqrt{1-x^2}}$$

$$\left.\begin{aligned} \frac{1}{3}\ln|y| &= \sin^{-1}x + C \\ \ln|y| &= 3\sin^{-1}x + C \end{aligned}\right\} \quad 0.5 \text{ point}$$

$$\xrightarrow[0.5\text{points}]{} \quad y = Ae^{3\sin^{-1}x} \longrightarrow \text{general solution.}$$

$$y(0) = 2 \longrightarrow 2 = Ae^{0} \Rightarrow A = 2$$

unique solution: $\quad$ 1 point

$$\boxed{y(x) = 2e^{3\sin^{-1}x}}$$

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Solution and Marking scheme.

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1.[4 points] (a) Consider the integral $\int_1^\infty \frac{3}{e^{3x}}\,dx$. Does it converge or diverge? If it converges, give its value.

$$\int_1^\infty \frac{3}{e^{3x}}\,dx = \lim_{t\to\infty}\int_1^t 3e^{-3x}\,dx = \lim_{t\to\infty}\left(-e^{-3x}\Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(-e^{-t} + e^{-3}\right) = e^{-3} \quad \text{converges.}$$

(b) Use the Comparison Test to determine if the integral $\int_0^1 \frac{4+2\sin(x)}{4x^4+\sqrt{x}}\,dx$ converges or diverges. If it converges, give an upper bound for its value.

$4+2\sin(x) \leqslant 6.$

As $x \mapsto 0$ $\quad 4x^4+\sqrt{x}$ behaves like $\sqrt{x}$.

For $0 \leqslant x \leqslant 1$ $\quad 4x^4+\sqrt{x} \geqslant \sqrt{x} \rightarrow \frac{1}{\sqrt{x}} \geqslant \frac{1}{4x^4+\sqrt{x}}$

so

$$\frac{4+2\sin(x)}{4x^4+\sqrt{x}} \leqslant \frac{6}{\sqrt{x}} \rightarrow \text{then } \int_0^1 \frac{4+2\sin(x)}{4x^4+\sqrt{x}}\,dx \leqslant \int_0^1 \frac{6}{\sqrt{x}}\,dx$$

converges

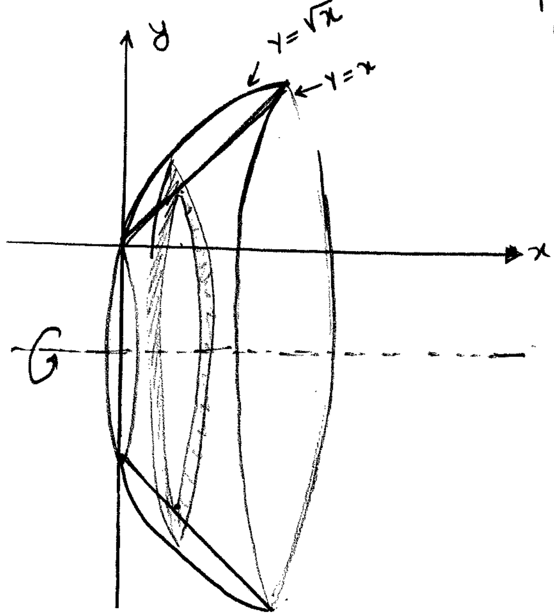
so does

$$\int_0^1 \frac{6}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{6}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 6\times 2\times\sqrt{x}\Big|_t^1 = \boxed{12}$$

upper bound.

converges.

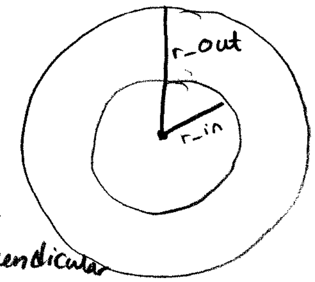
2.[4 points] Find the volume of the solid obtained by the curves $y = x$, $y = \sqrt{x}$, is rotated around the line $y = -2$. Include a sketch of the region and a typical cross-section of the solid in your solution.



points of intersection

$x = \sqrt{x}$

$\Rightarrow x = 0, 1$



cross-section area prependicular to x-axis is a washer.

$y = -2$

$r_{out} = \sqrt{x} + 2$

$r_{in} = x + 2$

$$A(x) = \pi (r_{out})^2 - \pi (r_{in})^2$$

$$= \pi \left(x + 4\sqrt{x} + 4 - x^2 - 4x - 4 \right)$$

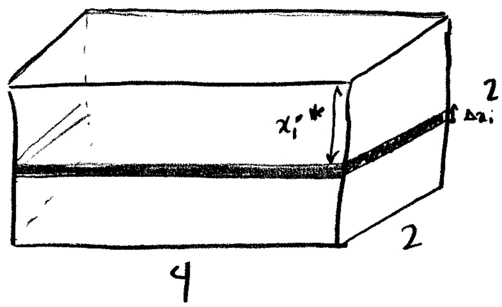
$$= \pi \left(-x^2 - 3x + 4\sqrt{x} \right)$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi \left(-x^2 - 3x + 4\sqrt{x} \right) dx$$

$$= \pi \left(\left(-\frac{x^3}{3} - \frac{3}{2}x^2 + \frac{8}{3} x^{3/2} \right) \Big|_0^1 \right)$$

$$= \pi \left(-\frac{1}{3} - \frac{3}{2} + \frac{8}{3} \right) = \boxed{\frac{5\pi}{6}}$$

3.[4 points] An aquarium 4 m long, 2 m wide, and 2 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. The density of water is $\rho = 1000\ \mathrm{kg/m^3}$ and the acceleration of gravity is $g = 9.8\ \mathrm{m/s^2}$. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.



1 point

$A_i = 4\times 2 = 8\ m^2$

$Vaulme_i = 8\,\Delta x\ m^3$

$m_i = V_i\,\rho = 8\,\Delta x\,\rho = 8000\,\Delta x$ kg

$F_i = m_i g = 8000\,\Delta x \cdot 9.8 = 78400\,\Delta x\ N$

1 point

n : the number of subintervals of length Δx

x_i^* : sample point in $[x_{i-1}, x_i]$

1 point

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 78400\, x_i^*\, \Delta x \quad \text{For } 0 \le x_i^* \le \tfrac{1}{2}$$

$$= \int_0^{\frac{1}{2}} 78400 x\, dx = 39200\, x^2 \Big|_0^{1/2} = 9800\ J$$

1 point

4.[3 points] Give the integral for the arclength of the curve $x = y^{3/2} + 2$ between $y = 0$ and $y = 2$ and then evaluate it.

$$f'(y) = \frac{3}{2} y^{1/2} \qquad f(y) = \frac{9}{4} y$$

$$L = \int_0^1 \sqrt{\left(\frac{9}{4}\right)y + 1} \; dy$$

$$= \frac{2}{3}\left(\frac{4}{9}\right)\left(\frac{9}{4}y + 1\right)^{3/2} \Big|_0^1$$

$$= \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right)$$

5.[3 points] Give the integral for the average value of the function $f(x) = xe^x$ on the interval $[1, 3]$ and then evaluate it.

$$A(x) = \frac{1}{3-1}\int_1^3 xe^x\,dx$$

$$= \frac{1}{2}\left[xe^x\Big|_1^3 - \int_1^3 e^x\,dx\right]$$

integration by parts

$x = u \rightarrow du = 1$

$e^x dx = dv$

$v = e^x$.

$$= \frac{1}{2}\left[3e^3 - e - e^3 + e\right)$$

$$= \frac{1}{2} \cdot 2e^3 = \boxed{e^3}$$

6.[2 points] Solve the initial value problem: $(1+x^2)\dfrac{dy}{dx} = 2y\,, \quad y(0) = 2\,.$

$$\frac{(1+x^2)}{dx} = \frac{2y}{dy} \qquad\qquad \frac{dx}{1+x^2} = \frac{dy}{2y}$$

$$\int \frac{dx}{1+x^2} = \int \frac{dy}{2y}$$

$$C + \arctan x = \frac{1}{2}\ln|y| \Rightarrow$$

$$\ln|y| = 2\arctan x + C \qquad |y| = e^{1/2 \arctan x + C}$$

$$\boxed{y = Ae^{2\arctan x}} \leftarrow \text{general solution.}$$

(0,2) initial point $\qquad 2 = Ae^0 \rightarrow \boxed{A = 2}$

unique solution:

$$\boxed{y(x) = 2e^{2\arctan x}}$$