6.4 Introduction to Sturm-Liouville Pbs.

- The eigenvalue problem

$$(p(x)u')' + q(x)u + \lambda r(x)u = 0$$

on the open interval $x \in (a,b)$
with

$$\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(b) + \delta u'(b) = 0 \end{cases}$$

is called a Sturm-Liouville problem provided
- $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous
- $p(x)$, $r(x) > 0$ in (a,b)

and
- $|\alpha| + |\beta| > 0$, $|\gamma| + |\delta| > 0$

- if $p(x)$ or $r(x)$ vanish at $x=a$ or $x=b$, or if the interval (a,b) is unbounded (i.e either a or $b \to \pm\infty$) then the problem is called a <u>singular</u> Sturm-Liouville problem; otherwise the problem is <u>regular</u>

- The function $r(x)$ is called the <u>weight function</u>

<u>Examples</u>

① $\begin{cases} \dfrac{d^2u}{dx^2} + \lambda u = 0 \\ u(0) = u(L) = 0 \end{cases}$ is a regular S-L problem with $p(x) = 1$ $q(x) = 0$ $r(x) = 1$

Bessel eq. ② $\begin{cases} x^2 \dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} + (x^2 - \nu^2)u = 0 & r \in (0, +\infty) \\ |u(0)| < +\infty \quad u(L) = 0 \end{cases}$ is a <u>singular</u> S L problem with $r(x) = +\frac{1}{x}$ $p(x) = x$ $q(x) = x$ $\lambda = -\nu^2$

- Note that we may also consider periodic S.L problems: where $u(a) = u(b)$ and $u'(a) = u'(b)$ are the BCs.

6.5 Properties of Sturm–Liouville problems (ODEs)

① Symmetry of the operator

Given two functions u and v satisfying

$$\begin{cases} \alpha v(a)+\beta v'(a)=0 \\ \gamma v(b)+\beta v'(b)=0 \end{cases} \qquad \begin{cases} \alpha u(a)+\beta u'(a)=0 \\ \gamma u(b)+\beta u'(b)=0 \end{cases}$$

then $$\int_a^b [u\mathcal{L}(v)-v\mathcal{L}(u)]\,dx = 0$$

Proof $$\int_a^b [u\mathcal{L}(v)-v\mathcal{L}(u)]\,dx$$

$$= \int_a^b \{u[(pv')'+qv]-v[(pu')'+qu]\}\,dx$$

$$= \int_a^b \{u(pv')'-v(pu')'\}\,dx$$

integrate by parts.

$$\left[upv'\right]_a^b - \int_a^b pu'v'\,dx - \left[pvu'\right]_a^b + \int_a^b pu'v'\,dx$$

$$= \left[p(uv'-vu')\right]_a^b$$

$$= p(b)\{u(b)v'(b)-v(b)u'(b)\} - p(a)\{u(a)v'(a)-v(a)u'(a)\}$$

$$= 0 \qquad \text{using the bcs.}$$

② Orthogonality of the eigenfunctions

Eigenfunctions corresponding to $\neq$ eigenvalues λ are orthogonal wrt the inner product

$$\langle u,v\rangle = \int_a^b u(x)v(x)r(x)\,dx.$$

Proof: Let u_n be an eigenfunction with λ_n e.value

u_m ———————— with λ_m e.value

$$\Rightarrow \begin{cases} \mathcal{L}(u_n) = -\lambda_n r u_n \\ \mathcal{L}(u_m) = -\lambda_m r u_m \end{cases}$$

then $\displaystyle\int_a^b \left[u_m \mathcal{L}(u_n) - u_n \mathcal{L}(u_m)\right] dx = 0$ by symmetry

$$= \int_a^b (\lambda_m - \lambda_n) r \, u_n u_m \, dx$$

$$= (\lambda_m - \lambda_n) \langle u_n, u_m \rangle$$

so unless $\lambda_m = \lambda_n$, $\langle u_n, u_m \rangle = 0$ $\square$

③ The eigenvalues of the Sturm-Liouville problem are real

Proof Suppose λ is a complex eigenvalue, corresponding to a complex solution u.

then $\mathcal{L}(u) = -\lambda r u = (pu')' + qu$

then taking the c.c on both sides $\Rightarrow$

$\mathcal{L}(u^*) = -\lambda^* r u^*$ $\Rightarrow$ λ^* is the eigenvalue corresponding to the eigenfunction u^*.

$\Rightarrow$ if $\lambda \notin \mathbb{R}$ then $\lambda \neq \lambda^*$ and so

$\langle u, u^* \rangle = 0$

But $\displaystyle\int_a^b u u^* r \, dx = \int_a^b |u|^2 r \, dx > 0$ unless u is identically 0.

$\rightarrow$ So we reach a contradiction, implying that $\lambda \in \mathbb{R}$.

(4) The eigenvalues of a regular Sturm Liouville problem are simple
i.e: if two functions have the same eigenvalue then these functions are linearly dependent.

Proof. let v_1 and v_2 be two eigenfunctions belonging to the same eigenvalue.

$$\mathcal{L}(v_1) = \lambda r v_1$$
$$\mathcal{L}(v_2) = \lambda r v_2$$

$$\Rightarrow \quad v_2 \mathcal{L}(v_1) = v_1 \mathcal{L}(v_2) = \lambda r v_1 v_2$$

so $v_2 \mathcal{L}(v_1) - v_1 \mathcal{L}(v_2) = 0$ for all x.

Recall that
$$v_2 \mathcal{L}(v_1) - v_1 \mathcal{L}(v_2) = v_2\left[(p v_1')' + q v_1\right] - v_1\left[(p v_2')' + q v_2\right]$$
$$= v_2 (p v_1')' - v_1 (p v_2')'$$
$$= \left(p \cdot (v_2 v_1' - v_1 v_2')\right)'$$

So $v_2 v_1' - v_1 v_2' = \text{constant}$

However, on the boundaries this quantity is 0
$\Rightarrow$
$$v_2 v_1' = v_1 v_2'$$
$$\Rightarrow \left(\frac{v_1}{v_2}\right)' = 0 \Rightarrow \boxed{v_1 = \alpha v_2}$$

(5) The set of all eigenvalues for a regular Sturm-Liouville problem forms an unbounded, strictly monotone sequence:
$$\lambda_0 < \lambda_1 < \lambda_2 \cdots < \lambda_n < \lambda_{n+1} < \cdots < +\infty.$$

and $\lim_{n \to \infty} \lambda_n = +\infty$; λ_0 is called the principal eigenvalue

(6) It is possible to construct a set of eigenfunctions $\{v_n\}$ of a regular Sturm-Liouville problem in such a way that

+ all eigenfunctions in the set are real
+ they are orthonormal w.r.t the inner product

$$\langle v_n, v_m \rangle = \int_a^b v_n(x) v_m(x)\, r(x)\, dx$$

+ the set is a <u>complete basis</u> for all piecewise continuous functions defined on the interval $[a,b]$, so that these functions can be written as the convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n v_n(x) \qquad \forall x \in [a,b].$$

with

$$a_n = \int_a^b f(x) v_n(x) r(x)\, dx.$$

(note: if v_n are not normalized, then

$$a_n = \frac{\int_a^b f(x) v_n(x) r(x) dx}{\int_a^b v_n^2(x) r(x) dx}.$$
)

<u>Definition</u>: We define the Rayleigh quotient

$$R(u) = -\frac{\int_a^b u \mathcal{L}(u)\, dx}{\int_a^b r(x) u^2\, dx}$$

(a functional of u).

Theorem : • The principal eigenvalue λ_0 of a regular Sturm- Liouville problem is the solution of

$$\lambda_0 = \inf_{u \in V} R(u) \qquad \text{(Rayleigh - Ritz formula)}$$

where V is the space of all continuous & differentiable functions on (a,b) such that u satisfy the BCs of the Sturm- Liouville problem, and $u \neq 0$ (not the trivial function)

• The function u_0 for which the minimum of $R(u)$ is achieved is the corresponding eigenfunction of the principal eigenvalue

Proof: let $\{\lambda_0 \ldots \lambda_n \ldots\}$ be the set of all eigenvalues of the S.L problem with $\{v_0 \ldots v_n \ldots\}$ the set of corresponding orthonormal eigenfunctions

then $$u = \sum_n a_n v_n(x)$$

and $$\mathcal{L}(u) = -\sum_n a_n \lambda_n r(x) v_n(x).$$

Then $$\int_a^b u \mathcal{L}(u) dx = \int_a^b -\sum_n \sum_m a_n a_m \lambda_n r(x) v_n(x) v_m(x) dx$$

$$= -\sum_n \sum_m \int a_n a_m \lambda_n r(x) v_n(x) v_m(x) dx$$

modulo some arguments about exchanging $\Sum$ and $\int$

$$= -\sum_n a_n^2 \lambda_n$$

$$\int_a^b r(x) u^2 dx = \int_a^b \sum_n \sum_m a_n a_m v_n(x) v_m(x) r(x) dx$$

same $$= \sum_n a_n^2$$

so $$R(u) = \frac{\sum_n a_n^2 \lambda_n}{\sum_n a_n^2}$$

now given that we know that $\forall n > 0$ $\lambda_n > \lambda_0$ then

$$R(u) \geqslant \frac{\lambda_0 \sum_n a_n^2}{\sum_n a_n^2} = \lambda_0$$

To have equality, we would require that $a_n = 0$ $\forall n > 0$ so that

$$R(u_0) = \frac{a_0^2 \lambda_0}{a_0^2} = \lambda_0$$

If $u_0 = a_0 v_0$ then u_0 is indeed the eigenfunction corresponding to the principle eigenvalue λ_0.

Notes ① given that

$$\int_a^b u \mathcal{L}(u)\,dx$$

$$= \int_a^b u\left[(p(x)u')' + q(x)u\right] dx$$

$$= \int_a^b q(x)u^2\,dx + \left[u p(x) u'\right]_a^b - \int_a^b p(x) u'^2\,dx$$

then
$$R(u) = \inf_{u \in V} \left[\frac{\int_a^b (p(x)u'^2 - q(x)u^2)\,dx - [uu'p]_a^b}{\int_a^b r u^2\,dx} \right]$$

This becomes particularly simple for Neumann or Dirichlet conditions: in that case

$$R(u) = \inf_{u \in V} \left[\frac{\int_a^b [p(x)u'^2 - q(x)u^2]\,dx}{\int_a^b r u^2\,dx} \right]$$

This form is often v. useful to determine the sign of λ_0, and to obtain order-of-magnitude estimates for it

(6.1) <u>Example 1</u>. Consider the S.L problem

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) - u'(0) = 0 \\ u(1) + u'(1) = 0 \end{cases} \qquad x \in [0,1].$$

Here, we have a S.L problem with

$$\begin{cases} p(x) = 1 \\ q(x) = 0 \\ r(x) = 1 \end{cases}$$

so $$R(u) = \frac{\int_0^1 u'^2\,dx - [u(1)u'(1) - u(0)u'(0)]}{\int_0^1 u^2\,dx}$$

but $u'(1) = -u(1)$ and $u'(0) = u(0) \Rightarrow [u(1)u'(1) - u(0)u'(0)] = -u(1)^2 - u(0)^2$
→ so clearly $R(u) \geq 0$ for all u, which proves that $\lambda_0 \geq 0$.

(6.4) <u>Example 2</u> Consider the S.L. problem.

$$\begin{cases} u'' + (\lambda - x^2)u = 0 \\ u'(0) = 0 \\ u(1) = 0 \end{cases}$$

We want to find an estimate for λ_0.

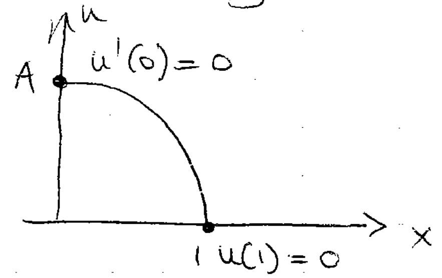
Here, we have the S.L problem with

$$\begin{cases} p(x) = 1 \\ q(x) = -x^2 \\ r(x) = 1. \end{cases} \qquad \text{so}$$

$$R(u) = \frac{\int_0^1 u'^2 + x^2u^2 \, dx - \cancel{[u(1)u'(1) - u(0)u'(0)]}^{\,0}}{\int_0^1 u^2 \, dx}$$

so we see that $R(u) \geq 0 \Rightarrow \lambda_0 \geq 0$

To obtain an estimate for λ_0, consider a trial function $u(x)$ that satisfy the boundary conditions



→ could try
$u(x) = A\cos\left(\frac{\pi}{2}x\right)$

or $u(x) = A(1-x^2)$

Using the 2nd option

$$R(u) = \frac{\int_0^1 \cancel{A^2}(-2x)^2 + x^2\cancel{A^2}(1-x^2)^2 \, dx}{\int_0^1 \cancel{A^2}(1-x^2)^2 dx}$$

$$= \frac{\int_0^1 (4x^2 + x^2 - 2x^4 + x^6)\, dx}{\int_0^1 1 - 2x^2 + x^4 \, dx}$$

$$= \frac{\frac{5}{3} - \frac{2}{5} + \frac{1}{7}}{1 - \frac{2}{3} + \frac{1}{5}} = \frac{37}{14} \quad \Rightarrow \quad 0 \leq \lambda_0 \leq \frac{37}{14}$$

Exercise; try the same procedure with $A\cos\left(\frac{\pi}{2}x\right)$

<u>Note</u>: True solution has $\lambda_0 = 2.597\ldots$ $\quad \frac{37}{14} = 2.64$

(2) It is actually possible to show that the sequence of eigenvalues of the S.L problem $(pu_n')' + qu_n = -\lambda_n r u_n$ is also the set of <u>all stationary pts</u> of the Rayleigh quotient $R(u)$ over V, and the eigenfunctions are the functions for which this stationary pt is achieved:

$$\lambda_n = R(v_n)$$

(7) Consider a regular S.L problem with eigenvalues $\{\lambda_0 \ldots \lambda_n \ldots\}$ and eigenfunctions $\{v_0 \ldots v_n \ldots\}$.
Then v_n has exactly <u>n roots</u> over the interval (a,b)

In particular, v_0 has no node within (a,b)

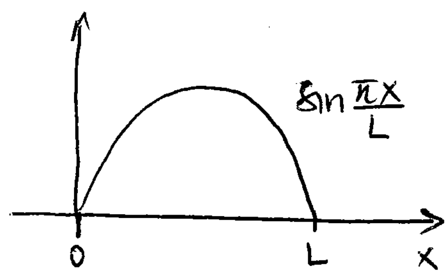
<u>Remark</u>: This is why the simplest guess for $u_0(x)$ for estimating λ_0 actually is also the best.

<u>Example</u>:

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = u(L) = 0 \end{cases}$$

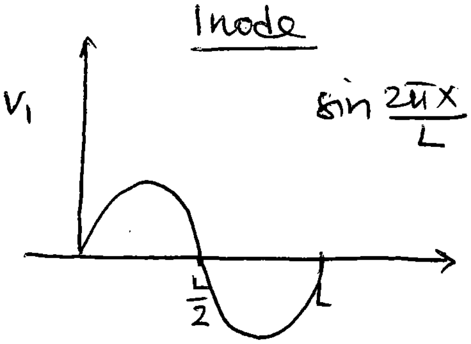
$$\begin{cases} \lambda_n = \dfrac{\pi^2}{L^2}(n+1)^2 \\ \\ v_n = \sin\left(\dfrac{\pi x}{L}(n+1)\right) \end{cases} \qquad n \geq 0$$

v_0:
0 nodes



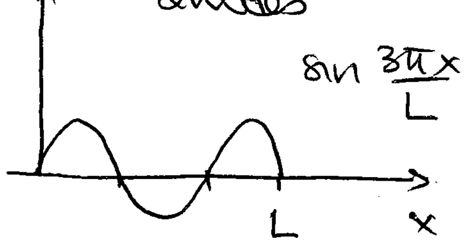
<u>1 node</u>

v_1



2 nodes

v_2



etc...

⑧ Asymptotic ($n \to \infty$) approximations to the eigenfunctions and eigenvalues of a regular SL problem

For large n, we know that $\lambda_n \to +\infty$; if this is the case, it is possible to approximate the eigenfunctions by

$$v_n(x) \simeq \frac{1}{(r(x)p(x))^{1/4}} \left\{ \alpha \cos\left[\sqrt{\lambda_n} \int_a^x \sqrt{\frac{r(x')}{p(x')}}\, dx'\right] + \beta \sin\left[\sqrt{\lambda_n} \int_a^x \sqrt{\frac{r(x')}{p(x')}}\, dx'\right]\right\}$$

(This formula is derived from the WKB approximation (see AMS 212))

In that case it's easy to see that

$$\sqrt{\lambda_n} \int_a^b \sqrt{\frac{r(x')}{p(x')}}\, dx' \simeq n\pi$$

$$\Rightarrow \qquad \lambda_n \simeq \left(\frac{n\pi}{\int_a^b \sqrt{\frac{r(x')}{p(x')}}\, dx'} \right)^2$$

Example
$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = u'(0) \\ u(1) = -u'(1) \end{cases}$$
we saw that $\lambda \geq 0$

This time, let's look for the eigensolutions:

$$u(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

to satisfy the bcs we calculate

$$u'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

so
$$\begin{cases} A = B\sqrt{\lambda} \\ A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda} = +A\sqrt{\lambda}\sin\sqrt{\lambda} - B\sqrt{\lambda}\cos\sqrt{\lambda} \end{cases}$$

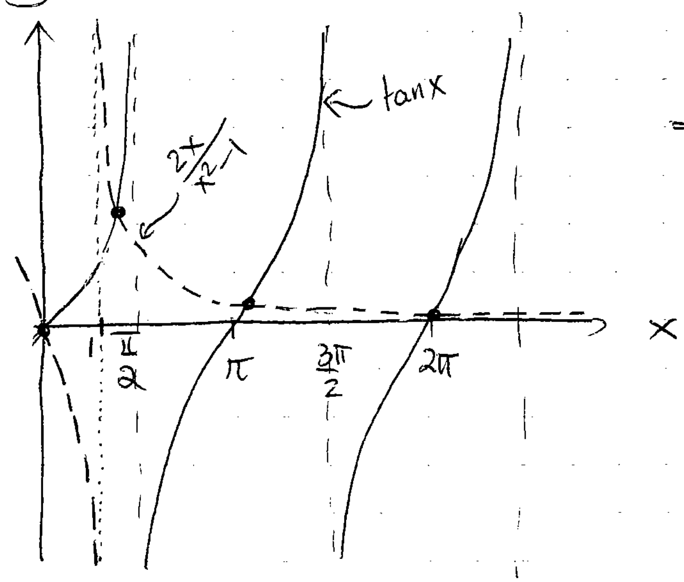
$$\Rightarrow \begin{cases} \sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda} = +\lambda\sin\sqrt{\lambda} - \sqrt{\lambda}\cos\sqrt{\lambda}. \\ A = B\sqrt{\lambda} \end{cases}$$

$$\Rightarrow \begin{cases} 2\sqrt{\lambda}\cos\sqrt{\lambda} = (\lambda-1)\sin\sqrt{\lambda} \\ A = B\sqrt{\lambda} \end{cases}$$

$$\Rightarrow \begin{cases} \tan\sqrt{\lambda} = \dfrac{2\sqrt{\lambda}}{\lambda-1} \\ A = B\sqrt{\lambda} \end{cases}$$

to find λ, we must solve the equation $\tan x = \dfrac{2x}{x^2-1}$

Graphically with $x > 0$



$\Rightarrow$ looks like

$x_n \simeq n\pi$

for large n

$\Rightarrow \lambda_n \simeq n^2\pi^2$

Check : using the asymptotic formula with

$r(x) = p(x) = 1 \quad q(x) = 0 \qquad a = 0, \; b = 1$

$\Rightarrow \quad \lambda_n \cong (n^2\pi^2)$. indeed

for large n.