Note: This general concept can also be applied to other PDES. The idea is derived from the physical concept of energy, but for most PDES $E(t)$ only needs to satisfy

$$\begin{cases} E(t) \geq 0 \\ \frac{dE}{dt} \leq 0 \\ E(t) \text{ depends on } u(x,t) \text{ and/or its derivatives.} \end{cases}$$

For the heat equation, for example,

$$\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \end{cases}$$

$\rightarrow$ use $E(t) = \frac{1}{2} \int_0^L u^2(x,t)\,dx$.

$\rightarrow$ homework: Prove that it has the required properties

CHAPTER 6 — Generalization of the method of separation of variables: Sturm-Liouville theory & eigenfunction expansions

In Chapter 4, we saw how some simple linear PDEs with very simple boundary conditions lend themselves to a solution method using separation of variables

We now generalize this concept to all second order homogeneous linear hyperbolic & parabolic PDEs of the form

$$u_t = \mathcal{L}_x^{(2)}(u)$$

or

$$u_{tt} = \mathcal{L}_x^{(2)}(u)$$

↑ a second order linear operator on u containing only spatial derivatives:

$$a(x)\, u_{xx} + b(x) u_x + c(x) u$$

Most of what we will see pertains to finite domains. We can now introduce a nomenclature for boundary conditions.

6.1 Classification of boundary conditions

Given the spatial domain Ω in which we are interested, there are typically three types of boundary conditions that can be applied at the edge of that domain ($\partial\Omega$). (Note: there are others too: as for example integral conditions).

(a) Dirichlet conditions

$$u(\underset{\sim}{r}, t) = f(\underset{\sim}{r}, t) \quad \text{where } \underset{\sim}{r} \in \partial\Omega$$

i.e. the value of the function is fixed on the contour (the edge) of the domain.

example. $u(\underset{\sim}{r},t) = 0$ (guitar string pinned @ end

$u(\underset{\sim}{r},t) = T_0$ (ends of a rod held at constant temperature T_0)

$u(\underset{\sim}{r},t) = \sin \pi t$ (end of a rope sinusoidally moved)

(b) Von Neumann conditions

$$\partial_\perp u(\underset{\sim}{r},t) = f(\underset{\sim}{r},t) \qquad \underset{\sim}{r} \in \partial\Omega$$

where $\partial_\perp$ denotes the derivative $\perp$ to the edge of the domain

i.e. the flux ~~of~~ u through the boundary is fixed on the contour ~~of~~ the domain

example $\frac{\partial u}{\partial z} = 0 \Rightarrow$ no heat flux through the boundary (insulating)

(c) Robin conditions

$$\alpha(\underset{\sim}{r},t)\partial_\perp u(\underset{\sim}{r},t) + \beta(\underset{\sim}{r},t)u(\underset{\sim}{r},t) = f(\underset{\sim}{r},t) \qquad \underset{\sim}{r} \in \partial\Omega$$

example:

- Matching to potential solutions $\nabla^2 u = 0$ outside of the domain.

Note: • This nomenclature applies to domains in any number of dimensions. (spatial)

• For a 1D interval, then

Dirichlet conditions on [a,b]: $\begin{cases} u(a,t) = u_1(t) \\ u(b,t) = u_2(t) \end{cases}$

Neumann conditions on [a,b]: $\begin{cases} \frac{\partial u}{\partial x}(a,t) = u_1(t) \\ \frac{\partial u}{\partial x}(b,t) = u_2(t) \end{cases}$

Robin conditions: $\begin{cases} \alpha u(a,t) + \beta u_x(a,t) = u_1(t) \\ \gamma u(b,t) + \delta u_x(b,t) = u_2(t) \end{cases}$

$|\alpha| + |\beta| > 0 \quad |\gamma| + |\delta| > 0.$

6.2 General form of parabolic & hyperbolic homogeneous linear equations supporting separation of variable

In the parabolic case

$$u_t = \frac{1}{r(x)m(t)}\left[(p(x)u_x)_x + q(x)u\right]$$

$$\begin{cases} \alpha u(a,t) + \beta u_x(a,t) = 0 \\ \gamma u(b,t) + \delta u_x(b,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

: homogeneous BCs (General Robin case) $(|\alpha|+|\beta|>0,\ |\gamma|+|\delta|\geq 0)$

: ICs.

In the hyperbolic case

$$u_{tt} = \frac{1}{r(x)m(t)}\left[(p(x)u_x)_x + q(x)u\right]$$

$$\begin{cases} \alpha u(a,t) + \beta u_x(a,t) = 0 \\ \gamma u(b,t) + \delta u_x(b,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x). \end{cases}$$

(same).

Note: it is always possible to transform any equation of the kind

$$\begin{cases} m(t)\, u_t = \mathcal{L}_x^{(2)}(u) \\ m(t)\, u_{tt} = \mathcal{L}_x^{(2)}(u) \end{cases}$$

in the above form

Say $\mathcal{L}_x^{(2)} = a(x)\,u_{xx} + b(x)u_x + c(x)u$ with $a \neq 0$

Multiply by $\dfrac{p(x)}{a(x)}$ with $p(x) = e^{\int \frac{b(x)}{a(x)}dx}$

then

$$\mathcal{L}_x^{(2)} = p(x)\,u_{xx} + \frac{p(x)}{a(x)}b(x)u_x + \frac{c(x)}{a(x)}p(x)u$$

$$= p(x)\,u_{xx} + \frac{dp}{dx}u_x + \frac{c(x)}{a(x)}p(x)u$$

$$= (pu_x)_x + \frac{c(x)}{a(x)}p(x)\,u.$$

since $\dfrac{dp}{dx} = \dfrac{b(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx} = \dfrac{b(x)}{a(x)}p(x)$

(And $r(x)$ is modified to absorb the additional $\frac{p(x)}{a(x)}$ factor).

6.3 Separation of variables

As usual let $u(x,t) = X(x)T(t)$ then

<u>Parabolic case</u>

$$m(t)\frac{\dot{T}}{T} = \frac{(p(x)X')' + q(x)X}{r(x)X}$$

so $m(t)\dot{T} = -\lambda T$

$$(p(x)X')' + q(x)X = -\lambda r(x)X$$

<u>Hyperbolic case</u>

$$m(t)\frac{\ddot{T}}{T} = \frac{(p(x)X')' + q(x)X}{r(x)X}$$

$m(t)\ddot{T} = -\lambda T$

$$(p(x)X')' + q(x)X = -\lambda r(x)X$$

The equation $(p(x)X')' + (q(x) + \lambda r(x))X = 0$

is a <u>Sturm-Liouville</u> equation

with bcs: $\begin{cases} \alpha X(a) + \beta X'(a) = 0 \\ \gamma X(b) + \delta X'(b) = 0 \end{cases}$

In what follows we will study the theory of Sturm-Liouville ODES.

Importance:

- It is an <u>eigenvalue equation</u>: it does not have to have solutions for all values of λ, only for a <u>discrete</u> set of λ_n; this set $\{\lambda_n\}$ is the spectrum of $\mathcal{L}$

- It is an eigenvalue problem in the sense of linear algebra since we are looking for $X(x)$ and $\tilde{\lambda}$ such that

$$L(X) = \tilde{\lambda} X \qquad (\tilde{\lambda} = -\lambda \text{ here})$$

with $$L(X) = \frac{1}{r(x)}\left[(p(x)X')' + q(x)X\right]$$

<u>Example</u> $$\frac{d^2u}{dx^2} + \lambda u = 0 \qquad u(0) = u(L) = 0$$

is a Sturm-Liouville problem with

$$p(x) = 1$$
$$q(x) = 0$$
$$r(x) = 1$$

The eigenvalues are $\lambda_n = \dfrac{n^2\pi^2}{L^2}$

and $X_n = \sin(\sqrt{\lambda_n}\, x)$

- The eigenfunctions of the problem determine the <u>spatial structure</u> of each <u>mode</u> of evolution

By expansion of the solution on the eigenmodes we can usually find an analytical expression.

- The eigenvalues of the problem entirely determine the temporal evolution of each mode.

Once λ is determined, we only have to solve

$$m(t)\,\dot{T} = -\lambda T \quad \text{in parabolic case}$$

$$m(t)\,\ddot{T} = -\lambda T \quad \text{in hyperbolic case}$$

This is particularly important when doing stability analyses of a system. Suppose their exists a steady state $\bar{u}(x)$ in a system. We want to study the evolution of the system under an initial small perturbation $\epsilon\xi(x)$ such that

$$u(x,0) = \bar{u}(x) + \epsilon\,\xi(x,0) \quad \text{at } t=0$$
$$u(x,t) = \bar{u}(x) + \epsilon\,\xi(x,t)$$

$\Rightarrow$ usually, we can write the evolution equation for ξ as a PDE leading to a Sturm-Liouville problem

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + \mathcal{L}_{\bar{u}}(\xi) = 0 \\[2ex] \text{or} \quad \dfrac{\partial^2 \xi}{\partial t^2} + \mathcal{L}_{\bar{u}}(\xi) = 0 \end{cases}$$

a linear operator which depends on $\bar{u}$, often a Sturm-Liouville operator

Then, by knowing the sign of the eigenvalues of $\mathcal{L}_{\bar{u}}$ we can determine the stability of the system.

<u>Definitions</u>
- The operator $\mathcal{L}(u) = (pu')' + qu$ is a Sturm-Liouville operator
- The function $r(x)$ is a weight function
- If $p(x)$ or $r(x)$ vanish at one of the endpoints of the interval considered <u>OR</u> if the interval is infinite / semi-infinite then the problem is called singular
- If not, then the problem is regular.