Topic 0: ALGEBRA

0.1: ABSOLUTE VALUE FUNCTION

The **absolute value function** simply turns all values positive.

For example, $|-4| = 4$

$|100| = 100$

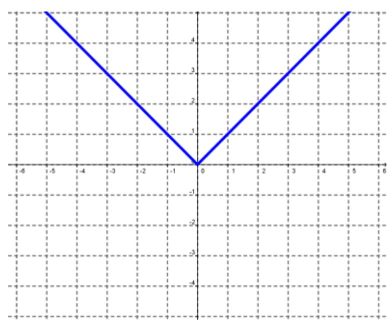
$\left|-\frac{3}{7}\right| = \frac{3}{7}$

$\left|3 - \sqrt{2}\right| = 3 - \sqrt{2}$

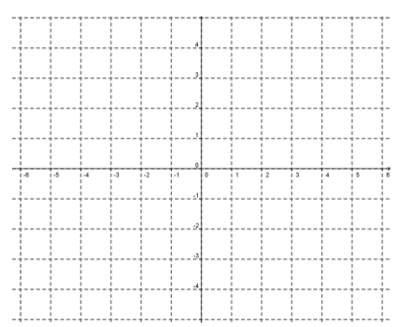
In fact, we can define the absolute value function $f(x) = |x|$ as a piecewise defined function as follows:

$$f(x) = |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

As a result, we can graph the absolute value function $f(x) = |x|$ as follows:



Evaluate a number of points for the function $f(x) = -|x+2| + 4$ and use them to sketch the graph.



0.2: EXPONENTS AND RADICALS

Exponents

An exponent indicates how many times a root is multiplied. In general, the expression a^n can be written as follows:

$$a^n = \underbrace{a \cdot a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$

For example, $5^3 = 5 \cdot 5 \cdot 5 = 125$

$$(-10)^4 = (-10)(-10)(-10)(-10) = 10{,}000$$

$$\left(\frac{3}{4}\right)^5 = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{243}{1024}$$

$$(2x+3)^2 = (2x+3)(2x+3) = 4x^2 + 12x + 9$$

Some rules:

(i) $a^0 = 1$ if a is any nonzero number

(ii) $a^{-n} = \frac{1}{a^n}$

(iii) $a^m a^n = a^{m+n}$

Show me!

$x^6 x^4 =$

(iv) $\frac{a^m}{a^n} = a^{m-n}$

Show me!

$\frac{6^5}{6^3} =$

(v) $(a^m)^n = a^{mn}$

Show me!

$(b^3)^4 =$

(vi) $(ab)^n = a^n b^n$

Show me!

$(2x)^5 =$

(vii) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

(viii) $a^{-1} = \frac{1}{a}$

(ix) $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

Show me!

$\left(\frac{x}{7}\right)^{-1} =$

(x) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Show me!

$\left(\frac{x}{y}\right)^{-6} =$

Simplify the following expressions:

$$\left(\frac{6x^3y^{-1}}{3x^2y^4}\right)^2 =$$

$$\frac{(3xy^3)^4x}{3(x^{-2}y^2)^{-1}} =$$

$$\left(\frac{5a}{b}\right)^{-3}(a^5b)^2 =$$

$$(3^2z^{-4})^{-1}xy^23^{-2}(xy^{-1}z)^3 =$$

Radicals

We can continue with the pattern of square roots and cube roots for any type root associated with a type of exponent. We call this "taking the n^{th} root."

$$\sqrt[n]{a} = b \text{ if and only if } b^n = a$$

NOTE: The n^{th} root of a product is equivalent to the product of the n^{th} root of each of the multiplied terms of the radicand. Put another way, if we take the n^{th} root of two things multiplied together, we can find it by multiplying the n^{th} roots of each of the multiplied things as represented by the general rule

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

Similarly, the n^{th} root of a quotient can be equivalently written as the quotient of the n^{th} root of the numerator and the nth root of the denominator, i.e.

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

For example, we can calculate roots as follows:

$$\sqrt[3]{-8x^3} = \sqrt[3]{-8 \cdot x^3} = \sqrt[3]{-8} \cdot \sqrt[3]{x^3} = -2 \cdot x = -2x$$

$$\sqrt[3]{-27{,}000x^6y^{12}} = \sqrt[3]{-27 \cdot 1{,}000 \cdot x^6 \cdot y^{12}} = \sqrt[3]{-27} \cdot \sqrt[3]{1{,}000} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{y^{12}} = -30x^2y^4$$

$$\sqrt{\frac{49}{144}} = \frac{\sqrt{49}}{\sqrt{144}} = \frac{7}{12}$$

Evaluate the following:

$\sqrt[3]{-64x^9} =$

$\sqrt[3]{125{,}000a^6b^{15}} =$

$\sqrt[3]{\frac{8}{z^{12}}} =$

$\sqrt[8]{\frac{a^8b^{24}}{c^{64}}} =$

When we are taking the nth root of a value, we can equivalently say that we are raising that value to the power of $1/n$.

For example, $\sqrt{x} =$

$\sqrt[3]{10} =$

$\sqrt[8]{xy^3} =$

$\sqrt[5]{\frac{a^5}{b^4}} =$

By using this notation, we can apply all of the laws that we previously learned about exponents.

We can also conclude that raising an expression to the n^{th} power and taking its n^{th} root will cancel out as follows:

$$\sqrt[n]{a^n} = (a^n)^{1/n} = a^{n/n} = a^1 = a$$

$$\left(\sqrt[n]{a}\right)^n = \left(a^{1/n}\right)^n = a^{n/n} = a^1 = a$$

For example, $\left(\sqrt[124]{5}\right)^{124} =$

$\sqrt[17]{\left(\frac{8x^4}{7y}\right)^{17}} =$

In general, any expression a raised to the power of m/n can be found by either raising a to the m^{th} power, then taking the n^{th} root OR by beginning by taking the n^{th} root and then raising the result to the m^{th} power.

Show me!

$\sqrt[n]{a^m} =$

$\left(\sqrt[n]{a}\right)^m =$

Calculate the following values:

$27^{-1/3} =$

$8^{4/3} =$

$\left(\frac{-1}{32}\right)^{3/5} =$

$10{,}000{,}000^{-2/7} =$

$\left(\frac{125}{64}\right)^{2/3} =$

Simplify the following expressions using rational exponents:

$\frac{x}{\sqrt{x}} =$

$\sqrt[3]{\frac{a^4 b^7 c^3}{a^{-2} b}} =$

$\frac{x y^4 \sqrt[9]{x^{18} y^3}}{y^5} =$

Since we know that the root of two multiplied terms is equivalent to the product of the root of each multiplied term of the radicand, we can occasionally reduce the radicand by removing perfect squares (squares of integers or of rational expressions) factors from beneath the root as follows:

$$\sqrt{27} = \sqrt{9 \cdot 3} = \sqrt{9}\sqrt{3} = 3\sqrt{3}$$

$$\sqrt{75x^5} = \sqrt{25 \cdot 3 \cdot x^2 \cdot x^2 \cdot x} = \sqrt{25}\sqrt{3}\sqrt{x^2}\sqrt{x^2}\sqrt{x} = 5 \cdot \sqrt{3} \cdot x \cdot x \cdot \sqrt{x} = 5x^2\sqrt{3x}$$

We say that a radical expression has been simplified once the remaining radicand has no perfect square factors left.

Simplify the following radical expressions:

$\sqrt{48} =$

$\sqrt[3]{81} =$

$\sqrt{500x^2} =$

$\sqrt{98x^3y^4z^9} =$

$\sqrt[3]{-8a^3b^7c} =$

However, since rational numbers do represent a specific value, we can combine them the way we would when dealing with variables (assuming that different irrational values are treated like different variables).

For example, $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$\sqrt{5} + 4\sqrt{5} =$

$3\sqrt[3]{10} + 5\sqrt[3]{10} - \sqrt[3]{10} =$

$\sqrt{7} + 10\sqrt{3} - \sqrt{3} + 6\sqrt{7} =$

However, since we can split up the root of multiplied terms, we do occasionally encounter terms that can be combined despite the fact that the radicand is different.

For example, $\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4}\sqrt{2} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$

Simplify the following:

$\sqrt{7} + \sqrt{28} =$

$\sqrt{20x} + \sqrt{5x} - \sqrt{125x} =$

$\sqrt{4x^4y} + x^2\sqrt{y} - x\sqrt{100x^2y} =$

$(2 + \sqrt{3})(4 + \sqrt{3}) =$

$(4 - 2\sqrt{2})^2 =$

<u>0.3: FACTORING POLYNOMIALS</u>

Factoring by Removing the Common Monomial (for any number of terms)

If all terms contain (i.e. are multiples of) a common term, we can remove that term through factoring. We call this **removing the common monomial**.

$$6ab^2 + 9b^2 - 300a^5b^3 - 15a^2b^4 = 3b^2(2a + 3 - 100a^5b - 5a^2b^2)$$

Factoring by Grouping (for an even number of terms, usually four)

Factoring by grouping consists of removing the common monomial from sets containing an equal number of terms. Provided that the remaining expressions from each set of terms are identical, the removed common monomials can then be combined into a single bracketed expression.

For example, the polynomial $5ab + 20b^2 - a - 4b$ can be factored by grouping as follows:

$$5ab + 20b^2 - a - 4b$$

$$5b(a + 4b) - 1(a + 4b)$$

$$(5b - 1)(a + 4b)$$

Factoring Trinomials (for three terms of the form $ax^2 + bx + c$)

Method:

(1) Find two values (let's call them A and B) whose sum is b and whose product is ac.

(2) Replace $ax^2 + bx + c$ with $ax^2 + Ax + Bx + c$. (This should be the same since we have $A + B = b$.)

(3) Use the new polynomial with four terms to factor by grouping.

For example, we could factor the trinomial $6x^2 + 5x - 4$ as follows:

$$6x^2 + 5x - 4$$

$$6x^2 + 8x - 3x - 4$$

$$2x(3x + 4) - 1(3x + 4)$$

$$(2x - 1)(3x + 4)$$

Factoring Monic Trinomials (for three terms of the for $x^2 + bx + c$)

Method:

(1) Find two numbers (let's call them A and B) whose sum is b and whose product is c.

(2) The factored form is $(x + A)(x + B)$.

Factoring Perfect Squares (for three terms of the form $a^2 + 2ab + b^2$)

We have the particular pattern that $a^2 + 2ab + b^2 = (a + b)^2$

For example, $9x^2 - 6x + 1 = (3x)^2 + 2(3x)(-1) + (-1)^2$

Factoring Using the Difference of Squares Pattern (for two terms of the form $a^2 - b^2$)

We have the particular pattern that $(a+b)(a-b) = a^2 - b^2$

For example, $25x^2 - 49 = (5x+7)(5x-7)$ since $25x^2$ is the square of $5x$ and 49 is the square of 7.

Factoring Using the Sum/Difference of Cubes Pattern (for two terms of the form $a^3 + b^3$ or $a^3 - b^3$)

Much like the difference of squares pattern exists and allows for easy factoring, there is also a distinct way to factor a sum or difference of cubes. It follows the pattern below:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

As a result, we need simply to apply the appropriate factoring pattern above for the sum or difference of two cubed expressions a and b. For example,

$x^3 + 8 = (x+2)(x^2 - 2x + 4)$ when we apply the first factoring pattern using $a = x$ and $b = 2$

$343 - 125k^3 = (7 - 5k)(49 + 35k + 25k^2)$ when we apply the second factoring pattern using $a = 7$ and $b = 5k$

Fully Factoring Expressions

To fully factor an expression:

(1) Remove the common monomial whenever possible.
(2) Factor the factors based on the number of terms:
    (i) 2 terms: Check for difference of squares, difference of cubes, sum of cubes
    (ii) 3 terms: Check for perfect squares, monic trinomials ($x^2 + bx + c$), non-monic trinomials ($ax^2 + bx + c$)
    (iii) 4 terms or more: Check for factoring by grouping.
(3) Verify that factors cannot be factored further. If they can, do it!

Factor the following:

$$27a^3b^8 - 45a^6b^3 + 63a^{10}b^2 + 72a^9b^2$$

$$4x^4 - 8x^3 + 6x - 12$$

$4x^2 - 8x + 3$

$-x^2 - 3x + 10$

$x^2 + 6x - 16$

$9z^2 - 66z + 121$

$100x^2 - 81$

$8x^3 - 27$

$2x^3 - 54$

$$3x^5 - 13x^4 - 10x^3$$

$$8x^3 - 40x^2 + 50x + 12x^2y - 10xy + 150y$$

$$20x^3 - 40x^2 - 45x + 90$$

$$-x^3 + 2x^2 + 48x$$

$$8rs^3 + 16s^3 + rt^6 + 2t^6$$

0.4: FRACTIONS AND RATIONALIZATION

Fractions

You may remember that when multiplying two fractions we would simply multiply the two numerators and multiply the two denominators in order to obtain the numerator and the denominator of the result, respectively. In fact, multiplying any two rational expressions is accomplished the same way. For example:

$$\frac{4x+4}{x+5}\cdot\frac{2x-1}{2x^2+2x}=\frac{4(x+1)}{x+5}\cdot\frac{2x-1}{2x(x+1)}=\frac{4(x+1)(2x-1)}{2x(x+5)(x+1)}=\frac{2(2x-1)}{x(x+5)}$$

NOTE: It helps to factor the numerators and denominators first when multiplying or dividing rational expression. That way, in order to multiply, we need only combine all of the factors into a single expression and simplification is easy.

Division by a rational expression is accomplished the same way division by a fraction is: we can change the division to a multiplication by the reciprocal (upside down rational expression). For example:

$$\frac{x^2-x}{2x^2+11x+15}\div\frac{3x^2+9x}{x-1}=\frac{x^2-x}{2x^2+11x+15}\cdot\frac{3x^2+9x}{x-1}$$

$$=\frac{x(x-1)}{(x+3)(2x+5)}\cdot\frac{3x(x+3)}{x-1}$$

$$=\frac{3x^2(x+3)(x-1)}{(x+3)(x-1)(2x+5)}$$

$$=\frac{3x^2}{2x+5}$$

Calculate the following multiplications and division of rational expressions:

$$\frac{x^2+6x+8}{x^2-16}\cdot\frac{-x^2+7x-12}{x^2-x-6}=$$

$$\frac{18x^2 + 24x}{10x + 1} \div \frac{9x + 12}{6x - 12} =$$

$$\frac{2x^2 + x - 1}{x - 4} \div 6x^2 + x - 2 =$$

Much like when dealing with simple fractions, if we plan on combining rational expression by adding or subtracting them, we must first find equivalent rational expressions with a common denominator. The general procedure is as follows:

To combine two rational expressions using addition/subtraction:

(1) Factor the denominators of all the terms.
(2) Find the Least Common Multiple (LCM) of the denominators.
(3) Write each term as an equivalent rational expression with the LCM as its denominator. This is accomplished by multiplying the top and bottom by the factors missing in the original denominators in order to achieve the LCM.
(4) Combine the rational expressions by adding the new numerators together and placing it over the common denominator.
(5) Simplify.

For example:

$$\frac{1}{x^2-5x+4}-\frac{x}{x^2+4x-32}=\frac{1}{(x-4)(x-1)}-\frac{x}{(x-4)(x+8)}$$

$$=\frac{1(x+8)}{(x-4)(x-1)(x+8)}-\frac{x(x-1)}{(x-4)(x-1)(x+8)}$$

$$=\frac{1(x+8)-x(x-1)}{(x-4)(x-1)(x+8)}$$

$$=\frac{-x^2+2x+8}{(x-4)(x-1)(x+8)}$$

$$=\frac{-(x^2-2x-8)}{(x-4)(x-1)(x+8)}$$

$$=\frac{-(x-4)(x+2)}{(x-4)(x-1)(x+8)}$$

$$=\frac{-(x+2)}{(x-1)(x+8)}$$

Calculate the following additions and subtractions of rational expressions:

$$\frac{9x+2}{3x^2-2x-8}+\frac{7}{3x^2+x-4}=$$

$$\frac{3}{x^2+6x+9}-\frac{1}{x-1}=$$

Complex fractions are rational expressions with rational expressions contained within the numerator and/or the denominator. For example, the expression $\frac{x+2+\frac{6}{x-3}}{x+\frac{1}{x-2}}$ is a complex fraction. This makes for some pretty ugly expressions, but they can be rewritten in a more user-friendly way as follows:

To simplify a complex fraction:

(1) Write the numerator as a single rational expression. Do the same for the denominator.
(2) Instead of dividing by the rational expression in the denominator, multiply the numerator by its reciprocal.
(3) Simplify.

As a result, the complex fraction mentioned above can be simplified as follows:

$$\frac{x+2+\frac{6}{x-3}}{x+\frac{1}{x-2}} = \frac{\frac{x+2}{1}+\frac{6}{x-3}}{\frac{x}{1}+\frac{1}{x-2}}$$

$$= \frac{\frac{(x+2)(x-3)}{x-3}+\frac{6}{x-3}}{\frac{x(x-2)}{x-2}+\frac{1}{x-2}}$$

$$= \frac{\left(\frac{(x+2)(x-3)+6}{x-3}\right)}{\frac{x(x-2)+1}{x-2}}$$

$$= \frac{\frac{x^2-x}{x-3}}{\frac{x^2-2x+1}{x-2}}$$

$$= \frac{x^2-x}{x-3} \cdot \frac{x-2}{x^2-2x+1}$$

$$= \frac{x(x-1)}{x-3} \cdot \frac{x-2}{(x-1)^2}$$

$$= \frac{x(x-2)}{(x-3)(x-1)}$$

Simplify the following complex fractions:

$$\frac{3 + \frac{9}{x-3}}{4 + \frac{12}{x-3}} =$$

$$\frac{\frac{x^2+1}{2x} - 1}{(x-1)^2} =$$

$$\frac{\frac{1}{2(x+h)} - \frac{1}{2x}}{h} =$$

Rationalization

Rationalizing an expression refers to writing it without using radicals. As a result, when you are asked to rationalize the denominator, we are asking you to rewrite a rational expression equivalently without making use of radicals in the denominator. Remember that when we multiply the top and bottom of a rational expression by the same value, the resulting rational expression is equivalent. So the question remains: By what expression do I need to multiply the top and bottom in order to get rid of the radical expression in the denominator?

Getting rid of a simple square root:

$$\frac{3}{\sqrt{5x}} \text{ can be rewritten as } \frac{3}{\sqrt{5x}} = \frac{3}{\sqrt{5x}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}} = \frac{3\sqrt{5x}}{\sqrt{5^2x^2}} = \frac{3\sqrt{5x}}{5x}$$

Getting rid of a simple cube root:

$$\frac{x}{\sqrt[3]{3x}} \text{ can be rewritten as } \frac{x}{\sqrt[3]{3x}} = \frac{3}{\sqrt[3]{3x}} \cdot \frac{\sqrt[3]{3^2x^2}}{\sqrt[3]{3^2x^2}} = \frac{3\sqrt[3]{3^2x^2}}{\sqrt[3]{3^3x^3}} = \frac{3\sqrt[3]{9x^2}}{3x} = \frac{\sqrt[3]{9x^2}}{x}$$

$$\frac{10}{\sqrt[3]{6x^2}} \text{ can be rewritten as } \frac{10}{\sqrt[3]{6x^2}} = \frac{10}{\sqrt[3]{6x^2}} \cdot \frac{\sqrt[3]{6^2x}}{\sqrt[3]{6^2x}} = \frac{10\sqrt[3]{6^2x}}{\sqrt[3]{6^3x^3}} = \frac{10\sqrt[3]{36x}}{6x} = \frac{5\sqrt[3]{36x}}{3x}$$

$$\frac{1+x}{\sqrt[3]{2x^5}} \text{ can be rewritten as } \frac{1+x}{\sqrt[3]{2x^5}} = \frac{1+x}{\sqrt[3]{2x^5}} \cdot \frac{\sqrt[3]{2^2x}}{\sqrt[3]{2^2x}} = \frac{(1+x)\sqrt[3]{2^2x}}{\sqrt[3]{2^3x^6}} = \frac{(1+x)\sqrt[3]{4x}}{2x^2}$$

Rationalize the denominator in the expressions below:

$$\frac{1}{\sqrt{7}} =$$

$$\frac{4}{3\sqrt{6a}} =$$

$$\frac{9x}{\sqrt[3]{2x}} =$$

$$\frac{2}{\sqrt[3]{10x^7}} =$$

$$\frac{x^3}{\sqrt[3]{2x^4y^2}} =$$

$$\frac{11}{\sqrt[4]{a^7bc^3}} =$$

If the denominator is a sum or difference of two terms containing square roots, we can use the difference of squares approach to eliminate the radicals. This consists of multiplying the top and bottom by the same two terms with the operation changed (i.e. if the denominator is an addition, multiply by a subtraction, and vice-versa).

Example: $\frac{1}{6+\sqrt{x}}$ can be written $\frac{1}{6+\sqrt{x}}=\frac{1}{6+\sqrt{x}}\cdot\frac{6-\sqrt{x}}{6-\sqrt{x}}=\frac{1(6-\sqrt{x})}{6^2-(\sqrt{x})^2}=\frac{6-\sqrt{x}}{36-x}$

$\frac{4x+3}{\sqrt{2x}-\sqrt{3}}$ can be written $\frac{4x+3}{\sqrt{2x}-\sqrt{3}}=\frac{4x+3}{\sqrt{2x}-\sqrt{3}}\cdot\frac{\sqrt{2x}+\sqrt{3}}{\sqrt{2x}+\sqrt{3}}=\frac{(4x+3)(\sqrt{2x}+\sqrt{3})}{(\sqrt{2x})^2-(\sqrt{3})^2}=\frac{4x\sqrt{2x}+4x\sqrt{3}+3\sqrt{2x}+3\sqrt{3}}{2x-3}$

Rationalize the denominators below:

$$\frac{-5}{1-\sqrt{6}}=$$

$$\frac{7+\sqrt{x}}{\sqrt{2}+\sqrt{x}}=$$

$$\frac{x^2}{x+\sqrt{3x}}=$$

$$\frac{7x-1}{1-\sqrt{2}}=$$

Rationalizing a numerator is accomplished the same way as rationalizing a denominator. We simply focus on eliminating the root from the top.