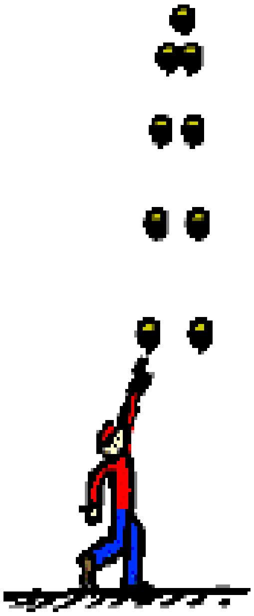
Topic 4: THE DERIVATIVE OF A FUNCTION

4.1: THE LIMIT DEFINITION OF THE DERIVATIVE AND THE SLOPE OF A GRAPH



You videotape your friend, Tim, who claims that he can throw a ball higher than anyone. The height is indeed impressive and you wonder at what speed he managed to launch the ball. Thanks to the video footage, you can tell how high the ball was at any given point in time and the following information is collected:

Time (s)	Height (m)
0.00	2.00000
0.25	6.69375
0.50	10.77500
0.75	14.24375
1.00	17.10000
1.25	19.34375
1.50	20.97500
1.75	21.99375
2.00	22.40000
2.25	22.19375

After much debate, you conclude that the best approximation you have (based on the table) is based on the speed in the first 0.25 seconds.

$$speed = \frac{change\ in\ height}{change\ in\ time}$$

So you approximate that Tim managed to throw the ball at a rate of

$$\frac{6.69375 - 2.00000}{0.25 - 0.00} \approx 18.775\ m/s$$

But if you had access to information about the height after 0.1 seconds, you would have used information about the first 0.1 seconds of the toss. However, knowing what the height of the ball had been after 0.01 seconds would have provided you with an even better approximation. And the information after 0.001 seconds would have been even more accurate of a measurement of the speed of the ball when Tim released it. Why? Because the shorter the interval of time you are using to approximate the speed of the ball is, the less time you leave for the speed of the ball to change (since it starts slowing down because of gravity as soon as Tim lets go).

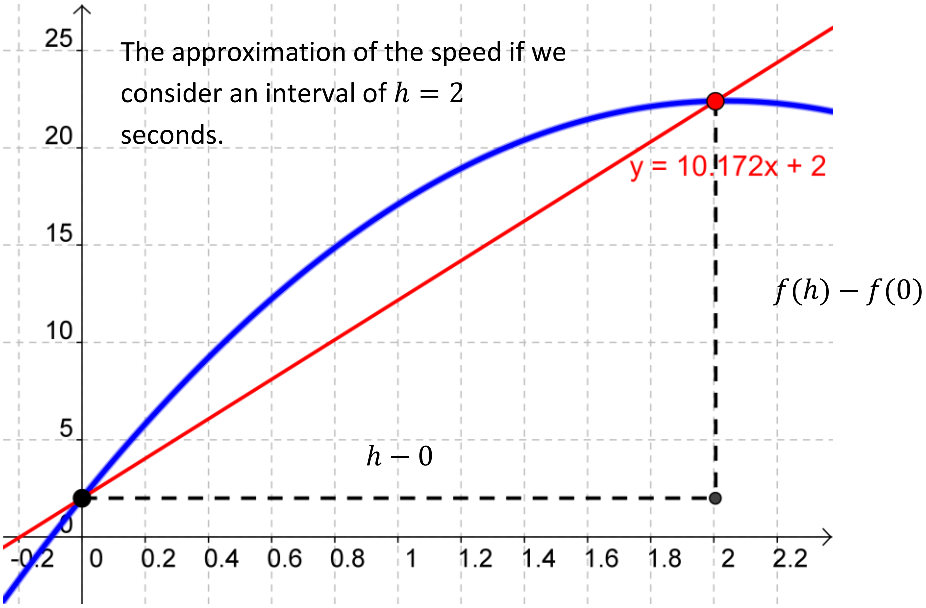
The general idea is this: Suppose we use an interval of h seconds for our approximation. The smallest possible interval h will provide the best approximation for the rate of change of the height (i.e. for the speed). So the exact speed would be the one that you would calculate if $h \to 0$.

If $f(x)$ was a function that represented the height of the ball (in meters) and x represented the time elapsed since the ball left Tim's hand (in seconds), you would find the exact speed of the ball when launched by performing the following calculation:

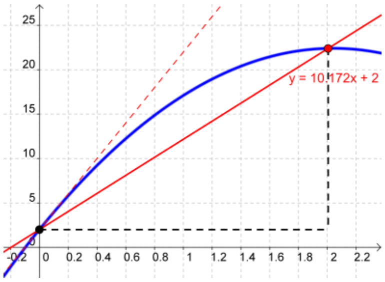
$$speed\ at\ time\ 0 = \frac{change\ in\ height}{change\ in\ time}\ \text{over the smallest interval of time } h \text{ possible}$$

$$= \lim_{h \to 0}\left(\frac{f(h) - f(0)}{h - 0}\right)$$

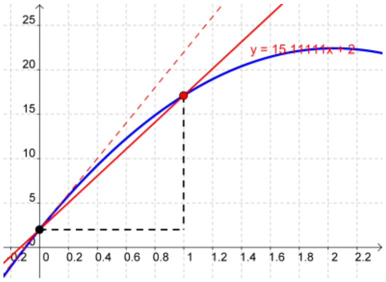
NOTE: If $f(x)$ represents the height of the ball and x represents the time, then the average speed (or "average rate of change of the height") is $\frac{(change\ in\ f(x))}{change\ in\ x} = \frac{rise}{run} = slope\ of\ the\ red\ line\ below$:


The approximation of the speed if we consider an interval of $h = 2$ seconds.

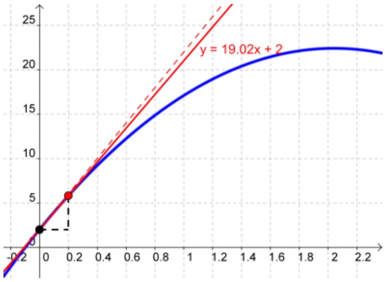
So the speed at time $x = 0$ would be the slope of the line if we let the interval become as small as possible (so the two points get infinitely close together). We call this line the **tangent line** (illustrated below by the dotted red line) and its slope is called the **derivative**.



$h = 2$



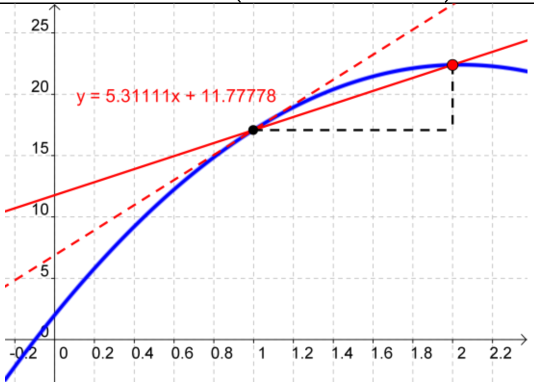
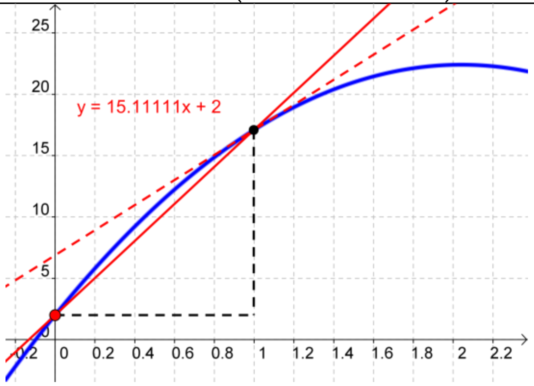
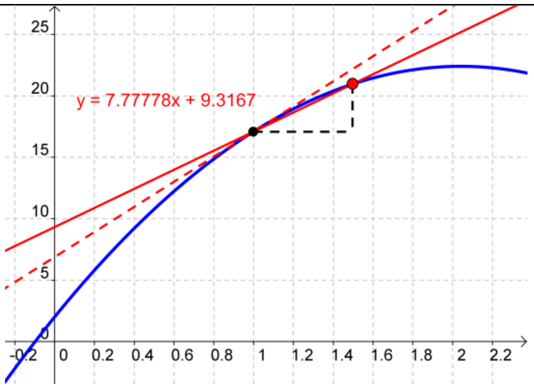
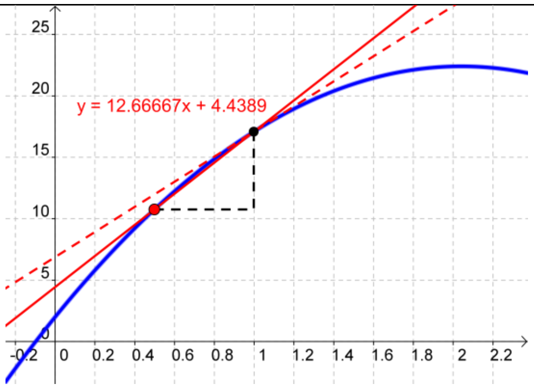
$h = 1$

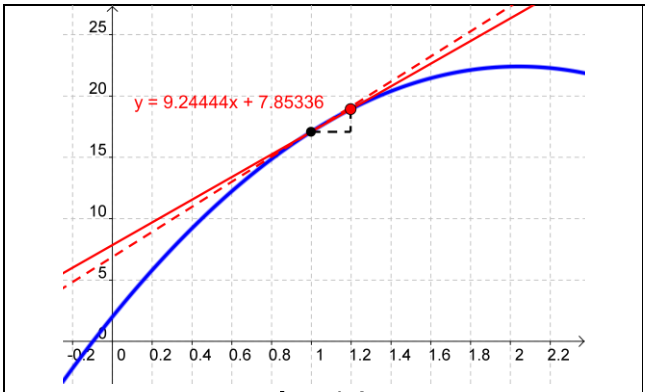


$h = 0.2$

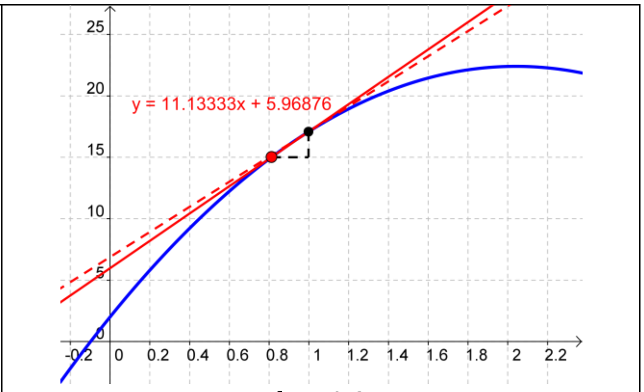
In this case, the derivative turns out to be $f'(0) = 20$. (The apostrophe indicates "derivative.")

We can use a similar method to find the speed of the ball after one second by constructing an interval of h seconds beginning or ending when $x = 1$ second and letting the interval become as small as possible for the best approximation ($h \to 0$).

Using an interval of time beginning at $x = 1$ sec	**Using an interval of time ending at $x = 1$ sec**
$f'(1) = \lim_{h \to 0} \left(\frac{f(1+h) - f(1)}{h} \right)$	$f'(1) = \lim_{h \to 0} \left(\frac{f(1) - f(1-h)}{h} \right)$
 $h = 1$	 $h = 1$
 $h = 0.5$	 $h = 0.5$



$h = 0.2$



$h = 0.2$

In both of the cases above, we would find the derivative is $f'(1) = 10.2$, so one second after the ball was released, it was travelling upward at a rate of 10.2 m/s.

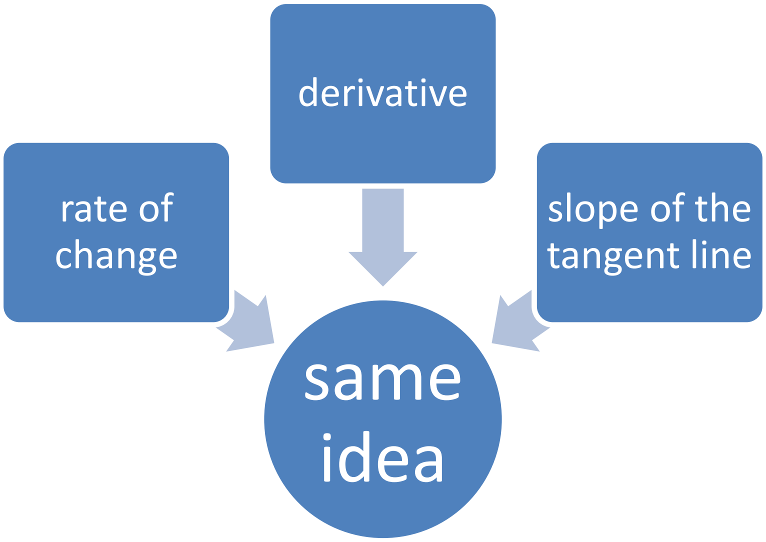
In general, the derivative of a function at the point $x = a$ can be calculated the following way:

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

OR we could define a derivative function $f'(x)$ which will calculate the derivative at any point by simply substituting the proper value of x into the resulting function:

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Derivative: The derivative of a function f at a value a (denoted $f'(a)$) is defined by the calculation $f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$ if the limit exists. It tells us how quickly $f(x)$ is changing as x changes at the point $(a, f(a))$, i.e. it gives us the instantaneous rate of change of $f(x)$ at a.



derivative

rate of change

slope of the tangent line

same idea

Tangent line: The tangent line of a curve $f(x)$ at a is the line whose rate of change (i.e. slope) is the same as that of the curve at a and which passes through the point $\big(a, f(a)\big)$. As a result, the slope of the tangent line corresponds to the derivative $f'(a)$.

Derivative function: Instead of observing the rate of change of a function f at a specific point, it is possible to create a function $f'(x)$ (also occasionally denoted $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$ or $\frac{dy}{dx}$ or y') that will determine the derivative of f at any point x by taking the following limit:

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

All or the notations below represent the derivative function of a function $y = f(x)$ with respect to the variable x:

$$y', \quad f', \quad f'(x), \quad \frac{dy}{dx}, \quad \frac{df}{dx}, \quad \frac{d}{dx}\big(f(x)\big), \quad D_x y$$

In the context of this course, we will primarily be using the notations $f'(x)$ and $\frac{df}{dx}$, but you should be able to recognize similar shorthand variations...

4.2: SOME RULES FOR DIFFERENTIATION

Differentiation: The act of finding the derivative.

Some key derivative functions:

(1) When $f(x) = x$.

$$\begin{aligned} f'(x) = \frac{df}{dx} &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{(x+h)-x}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{h}{h}\right) \\ &= \lim_{h\to 0}(1) \\ &= 1 \end{aligned}$$

(2) When $f(x) = x^2$.

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0}\left(\frac{(x+h)^2-x^2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(x^2+2xh+h^2)-x^2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2xh+h^2}{h}\right)$$

$$= \lim_{h\to 0}(2x+h)$$

$$= 2x$$

(3) When $f(x) = x^3$.

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0}\left(\frac{(x+h)^3-x^3}{h}\right)$$

(4) When $f(x) = \sqrt{x}$.

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$$

(5) When $f(x) = \frac{1}{x}$.

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0}\left(\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\right)$$

(6) When $f(x) = x^n$ for any nonzero value n, $f'(x) = nx^{n-1}$.

(7) When $f(x) = c$ where c is some real number, there is no change in the value of $f(x)$ as x changes, so $f'(x) = 0$.

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right) = \lim_{h \to 0}\left(\frac{c - c}{h}\right) = \lim_{h \to 0}\left(\frac{0}{h}\right) = 0$$

Find the derivative functions of the following:

$f(x) = x^{13}$

$f(x) = x^{-4}$

$f(x) = \sqrt[6]{x}$

$f(x) = -10$

$f(x) = \frac{1}{x^3}$

Derivative Laws:

(1) Constant Rule

$$\frac{d}{dx}(c) = 0 \text{ where } c \text{ is a real value.}$$

(2) Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1} \text{ where } n \text{ is a nonzero real value.}$$

(3) Constant Multiple Rule

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) \text{ where } c \text{ is a real value.}$$

(4) Sum Rule

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

(5) Difference Rule

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

NOTE: Multiplication and division are not so simple. We will see how in the section covering the **Product Rule** and the **Quotient Rule**.

Find the derivatives of the following functions:

$\frac{d}{dx}(x+2) =$

$\frac{d}{dx}(5x^3) =$

$\frac{d}{dx}(x^7 + 4x^2 + 2) =$

$\frac{d}{dx}\left(\frac{10}{x^4} + \sqrt{x}\right) =$

$\frac{d}{dx}\left(-3x + \frac{1}{x} + 4\sqrt[4]{x} - \sqrt[9]{x}\right) =$

$\frac{d}{dx}\left(-2\left(\frac{5}{\sqrt[3]{x}} - \frac{2}{x^6} + 10\right)\right) =$

$\frac{d}{dx}\left((x+2)(x^2-3)\right) =$

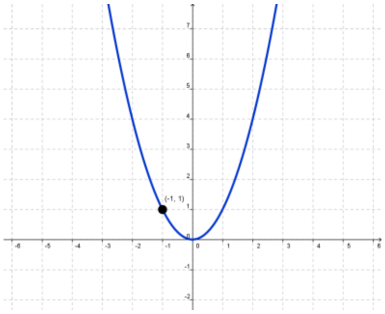
Finding the equation of the tangent line of a curve at a point a is now just a matter of algebra...

REMEMBER: The tangent line of a curve $f(x)$ at a is the line whose rate of change (i.e. slope) is the same as that of the curve at a and which passes through the point $(a, f(a))$. As a result, the slope of the tangent line corresponds to the derivative $f'(a)$.

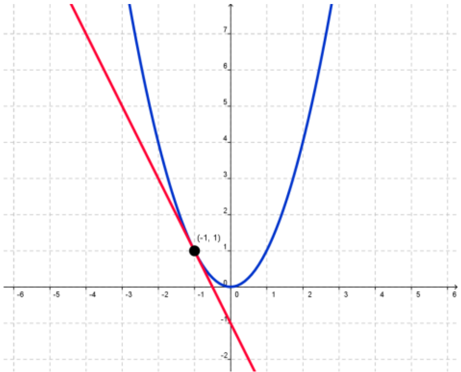
Start by finding the slope of the tangent line. This can be accomplished by finding the derivative function of the curve. We can then evaluate the derivative function at the point a.

Once we know the slope of the line, we can find the equation based on the fact that we know that the line must pass through the point $(a, f(a))$.

Let's say that we wanted to find the line tangent to the curve $f(x) = x^2$ at the point $x = -1$.



We start by finding the derivative function $f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(x^2) = 2x$. From here, we can establish that the slope of the tangent line at $x = -1$ is the derivative function evaluated at $x = -1$, i.e. the slope of the tangent line is $f'(-1) = 2(-1) = -2$.



Now, since we know that the slope of the line is -2, we can conclude that the equation of the tangent line is $y = -2x + b$ for some value b. We also know that the line must pass through the point $\left(-1, f(-1)\right) = (-1, (-1)^2) = (-1,1)$. So if the line passes through the point $(-1,1)$, we know that the equation must work when $x = -1$ and $y = 1$. From there, we can use algebra to figure out the b-value.

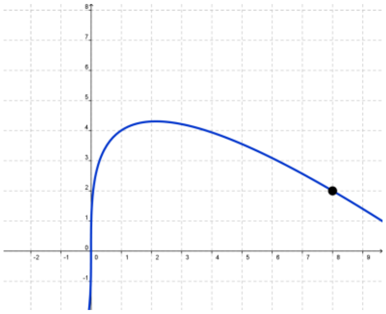
$$1 = -2(-1) + b$$

$$1 = 2 + b$$

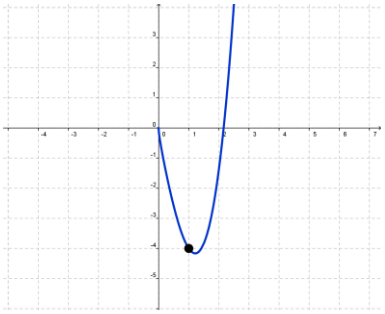
$$b = -1$$

So the equation of the tangent line is $y = -2x - 1$.

Find the equation of the line tangent to the curve $f(x) = 5\sqrt[3]{x} - x$ at the point $x = 8$, then sketch the line on the graph.



Find the equation of the line tangent to the curve $f(x) = x^3 - 4x - \sqrt{x}$ at the point $x = 1$, then sketch the line on the graph.



When the derivative of a function $f(x)$ can be calculated at a point a, we say that $\boldsymbol{f}$ **is differentiable at** $\boldsymbol{a}$. If we can calculate the derivative of a function $f(x)$ at any point on an interval I, we say that $\boldsymbol{f}$ **is differentiable on the interval** $\boldsymbol{I}$.

When we are faced with finding out when a function $f(x)$ is differentiable, we can accomplish this by finding the domain of $f'(x)$. (If we can calculate the value of $f'(a)$, then we can calculate the derivative of $f(x)$ at a.)

Find the intervals over which the following functions are differentiable.

$f(x) = 2x^2 + 3x + 6$

$f(x) = \sqrt[3]{x}$

$f(x) = -3\sqrt{x} + 2$

$f(x) = \frac{3}{x}$

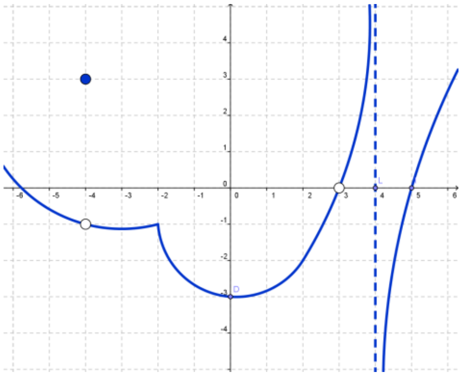
$f(x) = \frac{1}{x^3} - x^{7/4} + 1$

Graphically, we can determine when the function $f(x)$ represented by the curve is NOT differentiable when at points where any of the following occurs:

(1) The function is not defined, i.e. the point is not in the domain of f. This is usually manifested as a vertical asymptote or a hollow point.

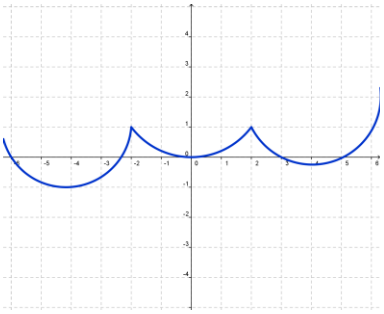
(2) The curve is discontinuous.

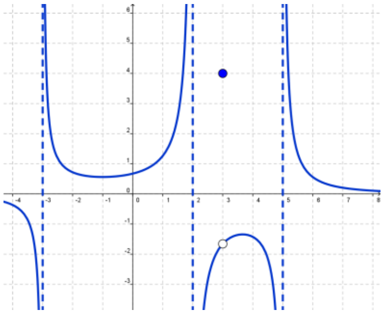
(3) The curve is not smooth, i.e. it makes an angle.

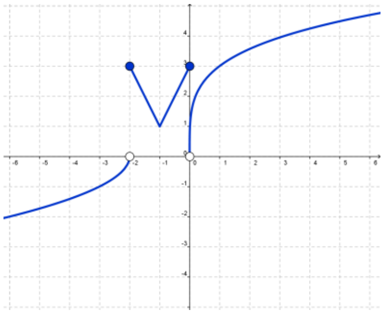


For example, in the graph above, the function is not differentiable at the points $x = -4, -2, 3$, and 4. As a result, the function is differentiable on the intervals $(-\infty, -4)$, $(-4, -2)$, $(-2,3)$, $(3,4)$, and $(4, \infty)$.

Find the intervals over which the functions illustrated in the graphs below are differentiable.



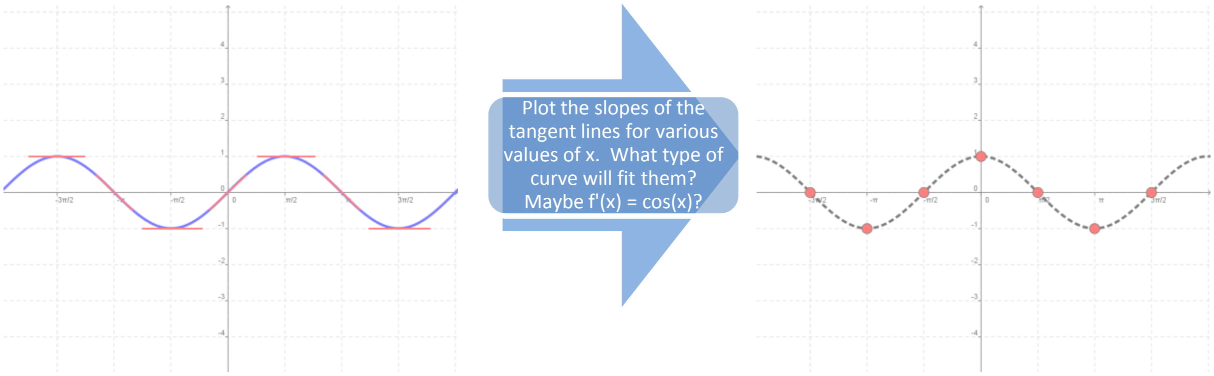




So far, we know how to take the derivatives of polynomials and of radical functions, but since the world does not solely revolve around equations of this nature, there are many more to learn!

Derivatives of sine and cosine:

The sine function is cyclical (i.e. it repeats itself), so we can assume that its derivative will also be a cyclical function. But what would it look like? Let's try to form an educated guess based on the derivatives of the $f(x) = \sin x$ curve at a few points. (Remember that the derivative of a function at a point corresponds to the slope of the tangent line at that point.)

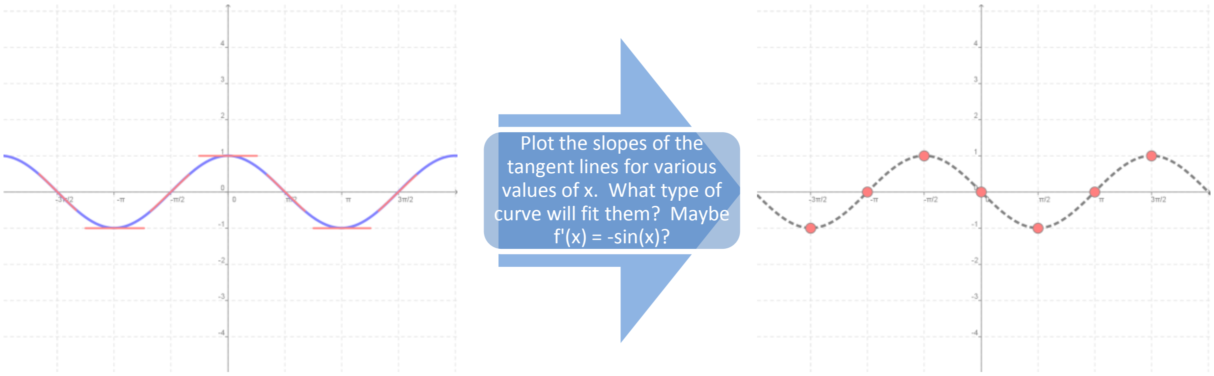


Plot the slopes of the tangent lines for various values of x. What type of curve will fit them? Maybe f'(x) = cos(x)?

We could hypothesize that

$$\frac{d}{dx}(\sin(x)) =$$

But what about the derivative of $f(x) = \cos x$? Let's use a similar approach to obtain a hypothesis:



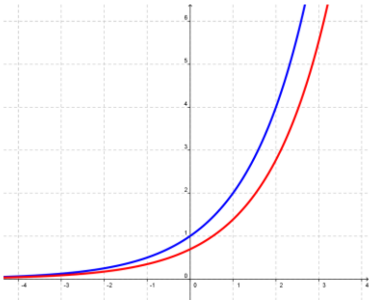
Plot the slopes of the tangent lines for various values of x. What type of curve will fit them? Maybe f'(x) = -sin(x)?

We could hypothesize that

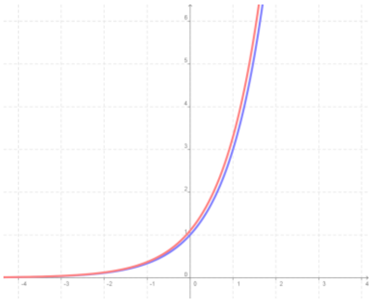
$$\frac{d}{dx}(\cos(x)) =$$

The number *e*: It can be observed that exponential functions such as $f(x) = 2^x$ and $g(x) = 3^x$ have derivative values that are directly proportional to the value of the original function...

$f(x) = 2^x \quad f'(x) \approx 0.6931 \cdot f(x)$



$g(x) = 3^x \quad g'(x) \approx 1.0986 \cdot g(x)$



But if we choose a specific value for our base somewhere in between 2 and 3, we obtain an exponential function whose derivative is exactly the same as the original function. We call this value $e \approx 2.71828$ and we have the following derivative:

$$\frac{d}{dx}(e^x) = e^x$$

4.3: THE PRODUCT AND QUOTIENT RULES

Product Rule:

If f and g are both differentiable, then we can find the derivative of their product as follows:

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)]$$

Some refer to the Product Rule as the "Travelling Prime Rule" because if we use alternate notation, we get

$$(fg)' = f'g + fg'$$

$$(fgh)' = f'gh + fg'h + fgh'$$

$$(fghl)' = f'ghl + fg'hl + fgh'l + fghl'$$

etc., for differentiable functions f, g, h, and l.

Find the derivatives of the following functions:

$$f(x) = x^2 e^x$$

$f(x) = (3x + 1)\sin(x)$

$$f(x) = \left(\sqrt{x} - 3\sqrt[3]{x}\right)(4e^x - 10)$$

$$f(x) = \frac{1}{x}(x^5 + e^x)$$

$$f(x) = \cos^2(x)$$

Quotient Rule:

If f and g are differentiable, then we can find the derivative of their quotient to be

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

In other words (using cleaner notation),

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

For example, if we wished to calculate the derivative of the function $F(x) = \frac{\cos(x)}{2x}$, we could employ the Quotient Rule as follows:

$$\frac{d}{dx}\left(\frac{\cos x}{2x}\right) = \frac{\frac{d}{dx}(\cos x)\cdot 2x - \cos x\cdot\frac{d}{dx}(2x)}{(2x)^2}$$

$$= \frac{(-\sin x)(2x)-(\cos x)(2)}{(2x)^2}$$

$$= \frac{-2x\sin x - 2\cos x}{4x^2}$$

$$= \frac{-x\sin x - \cos x}{2x^2}$$

Find the derivatives of the functions below:

$$f(x) = \frac{e^x + 1}{x^2}$$

$$f(x) = \frac{4x^2 + 9}{\sin(x)}$$

$$f(x) = \frac{6\cos(x)}{2e^x}$$

$$f(x) = \frac{2x^6 e^x}{x+1}$$

$$f(x) = \frac{4 - x}{3\sin(x)}$$

4.4: DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

These new rules open the door to finding even MORE derivatives! For example, if we revisit trigonometric functions, we can now find the derivative of the tangent function as follows:

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] \\
&= \frac{\frac{d}{dx}(\sin x)\cdot\cos x - \sin x\cdot\frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

Using the rules with which we are familiar, combined with the derivatives of trigonometric functions and various trigonometric properties, we can obtain a solid list of derivatives of the main trigonometric functions:

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

Ms. Lefebvre's memory tricks for learning trigonometric derivatives:

(1) If you want the derivative of a trigonometric function that begins with the letter "c" ($\frac{d}{dx}\cos(x)$, $\frac{d}{dx}\csc(x)$, $\frac{d}{dx}\cot(x)$) there should be a " $-$ " in front.

(2) The "messy derivative" table:
Find the trig function whose derivative you want in the table below. The product of the other three items in that row make up the derivative.

$+$	$\sec(x)$	$\sec(x)$	$\tan(x)$
$-$	$\csc(x)$	$\csc(x)$	$\cot(x)$

Find the derivative of the function

$$f(x) = \frac{\tan(x)}{\sec(x)}$$

4.5: THE CHAIN RULE

Chain Rule:

If f and u are both differentiable and $F = f \circ u = f\big(u(x)\big)$, then F is differentiable and its derivative function F' is given by the product below:

$$F'(x) = \frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = f'\big(u(x)\big) \cdot u'(x)$$

In the equation above, $\frac{dF}{dg}$ refers to the value of the derivative of $F(x)$ if we treat the inner function $g(x)$ as a simple variable. The term $\frac{dg}{dx}$ is simply the derivative with respect to x of the inner function.

For example, the function $F(x) = \sin(3x^2)$ will vary at a rate different from those of the functions $f(x) = \sin(x)$ and $g(x) = 3x^2$, but we know how to differentiate each of these functions independently and we have that $F(x) = f\big(u(x)\big)$. We can therefore apply the Chain Rule as follows:

$$\begin{aligned}
\frac{d}{dx}\big(F(x)\big) &= \frac{dF}{du} \cdot \frac{du}{dx} \\
&= (\sin u)' \cdot u' && \text{if we let } u = 3x^2 \\
&= \cos u \cdot u' \\
&= \cos(3x^2) \cdot 6x \\
&= 6x \cos(3x^2)
\end{aligned}$$

Find the derivatives of the following functions:

$f(x) = e^{2x+3}$

$f(x) = \csc\,(4x^2 + x)$

$f(x) = \left(\frac{1}{\sqrt{x}} - 2\right)^3$

$f(x) = \sin^2\left(\frac{7}{x}\right)$

$f(x) = e^{3\cot(x)}$

4.6: DERIVATIVES OF EXPONENTIAL FUNCTIONS

Now, if we use the Chain Rule, we could prove that $\frac{d}{dx}(a^x) = a^x \ln a$ as follows:

$$\begin{aligned}
\frac{d}{dx}(a^x) &= \frac{d}{dx}\left(\left(e^{\ln a}\right)^x\right) \\
&= \frac{d}{dx}\left(e^{\ln a \cdot x}\right) \\
&= \frac{d}{dg}(e^g) \cdot \frac{d}{dx}(\ln a \cdot x) && \text{if we let } g = \ln a \cdot x \\
&= e^g \cdot \ln a \\
&= e^{\ln a \cdot x} \ln a \\
&= \left(e^{\ln a}\right)^x \ln a \\
&= a^x \ln a
\end{aligned}$$

So now we can identify the mysterious constant by which an exponential function $f(x) = a^x$ is multiplied in order to obtain the derivative: it's $\ln a$!

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Find the derivative of the following functions:

$f(x) = 5^x$

$f(x) = \tan(3^x + 1)$

$f(x) = \sin(\csc(\pi^x))$

$f(x) = \frac{e + 2\tan(x)}{\cos x}$

$f(x) = \cot(4^x) + \frac{1}{7\sin(x)}$

4.7: HIGHER ORDER DERIVATIVES

We've already seen that the rate of change of the position of an object as time changes corresponds the the velocity of the object. But what if we wanted to know the rate of change of the velocity as time changes (i.e. the object's acceleration)? We need only derive the derivative function!

For example, if a particle's distance in meters from its initial resting point increases according to the function $f(t) = t^2$, where t is the time in seconds, then we know that its velocity is increasing according to the function $f'(t) = 2t$. Moreover, we know that the acceleration (rate of change of velocity) is $f''(t) = 2$.

This idea can be applied as many times as desired; we call the resulting **functions higher order derivatives** and they can be denoted the following ways:

1st order derivative:	$f'(x)$	$f^{(1)}(x)$	$\frac{dy}{dx}$
2nd order derivative:	$f''(x)$	$f^{(2)}(x)$	$\frac{d^2y}{dx^2}$
3rd order derivative:	$f'''(x)$	$f^{(3)}(x)$	$\frac{d^3y}{dx^3}$
	Etc.	Etc.	Etc.

Find the first, second, and third order derivatives of the function $f(x) = (2 - 3x)^{-1/2}$.

Find the first second and third order derivatives of the function $f(x) = xe^{4x}$.

Let $f(x) = e^{4x+6}$. Find the first, second, and third order derivatives of f, then create a formula for calculating the n^{th} order derivative of f.

4.8: IMPLICIT DIFFERENTIATION

Occasionally, the relationship between two variables x and y becomes a complicated matter. They're values are intertwined using complicated rules and it becomes quite difficult to disentangle the two and express the relationship explicitly as we usually do ($y = f(x)$) because there is no simple way to isolate y. However, the very fact that there exist values that respect the relationship described tells us that the relationship can be graphed. And if the function can be graphed, then there exists a tangent line to the curve at each point and, consequently, a derivative corresponding to the slope of that tangent line. In these situations we employ what is commonly referred to as "**implicit differentiation**." This method consists of the following steps:

(1) Differentiate both sides with respect to x, using the Chain Rule to differentiate y with respect to x.

(2) Isolate the expression $\frac{dy}{dx}$ which results from the Chain Rule in order to express the derivative of y explicitly in terms of x and y.

(3) Determine the value of the derivative at any point on the curve by plugging in the coordinates of the point as your x and y values.

For example, if we needed to differentiate y with respect to x in the relationship dictated by the equation $x^3 + y^3 = 3xy$ (this is called the "folium of Descartes"). If we plot the points for which this relationship holds, we obtain the following graph on the right. But what if we wanted to find the tangent to the curve at the point $(1.5, 1.5)$, we could do so as follows:

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3xy)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(3xy)$$

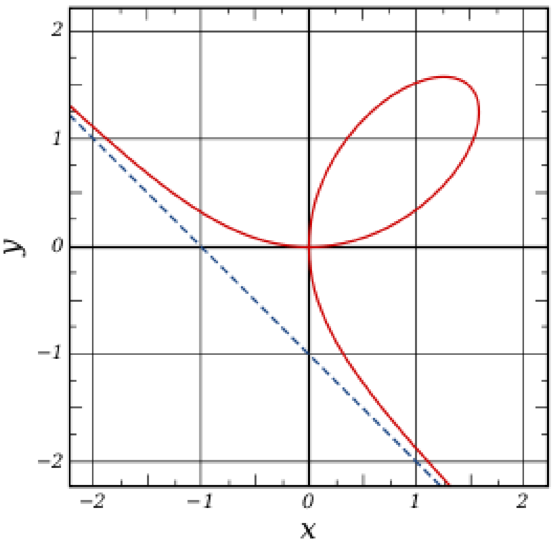
$$3x^2 + \frac{d}{dy}(y^3)\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$ [*]

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$$



This equality will allow us to find the slope of the line tangent to the curve provided that we know both the x- and y-coordinates of the point at which we are looking for the tangent line. (Note that before, we would be able to find the slope of the tangent line based on the value of x-alone.)

So if we wish to find the line tangent to the curve at the point $(1.5, 1.5)$, we know that the slope is

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x} = \frac{(1.5) - (1.5)^2}{(1.5)^2 - (1.5)} = -1$$

Also, we know the the line will pass through the point $(1.5, 1.5)$, so we can deduce the following:

$$y = -x + b$$

$$1.5 = -1.5 + b$$

$$b = 3$$

So the line tangent to the curve at $(1.5, 1.5)$ has the equation $y = -x + 3$.

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[*] This step incorporates the use of the Chain Rule when differentiating the second term on the left hand side of the equation, as well as the Product Rule when differentiating the right hand side of the equation.

Find the equation of the line tangent to the curve $y^2 = x^3(2 - x)$ at the point (1, 1).

Find the equation of the tangent line to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(3, 1)$.

Find the equation of the tangent line to the curve $2y^3 + y\ln(x) = -17 + x^2$ at the point $(1, -2)$.

4.9: DERIVATIVES OF LOGARITHMIC FUNCTIONS

We can also use implicit differentiation to determine the derivative of a logarithmic function. Suppose we want to find the derivative of the function $y = \log_a(x)$. Then we have that $a^y = x$.

$$\frac{d}{dx}a^y = \frac{d}{dx}x$$

$$\frac{d}{dy}a^y\frac{dy}{dx} = 1$$

$$a^y\ln(a)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a^y\ln(a)}$$

$$\frac{dy}{dx} = \frac{1}{x\ln(a)}$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Find the derivative function of the following:

$f(x) = \log_6(x)$

$f(x) = \ln(5\sin(x^2))$

$f(x) = \log(8x + e^{2x})$

4.10: LOGARITHMIC DIFFERENTIATION

We can now combine our knowledge of logarithms, of their derivatives, and of implicit differentiation in order to perform what is called **logarithmic differentiation**. This can be used to take the derivatives of a series of multiplied and divided expressions with more ease than by using the quotient, product, and chain rules outright.

For example, if we wish to find the derivative of the function $f(x) = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$ we can proceed as follows:

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

$$\ln(y) = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

$$\ln(y) = \ln\left(x^{3/4}\right) + \ln\left(\sqrt{x^2+1}\right) - \ln\left((3x+2)^5\right)$$

$$\ln(y) = \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left(\frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right)$$

$$\frac{d}{dy}\ln(y)\frac{dy}{dx} = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{1}{x^2+1}(2x) - 5\frac{1}{3x+2}(3)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$$

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

In general, logarithmic differentiation follows three steps:

(1) Take the natural logarithms of both sides of an equation and use the Laws of Logarithms to simplify.

(2) Differentiate implicitly with respect to x.

(3) Solve the resulting equation for $\frac{dy}{dx}$.

Find the derivative of the function $f(x) = \sqrt{x}e^{x^2}(x^2+1)^{10}$ using logarithmic differentiation.

Find the derivative of the function $f(x) = \frac{\sin^2(x)\tan^4(x)}{(x^2+1)^2}$ using logarithmic differentiation.

Another time when logarithmic differentiation comes in handy is when both the base AND the exponent of an exponential function contain the variable. Since we cannot choose between applying the rule $\frac{d}{dx}(a^x) = a^x \ln(a)$ and $\frac{d}{dx}(x^n) = nx^{n-1}$, we have to determine a new way of differentiating such functions.

For example, if we needed to find the derivative of the function $f(x) = x^x$, we would be able to do so as follows:

$$y = x^x$$

$$\ln(y) = \ln(x^x)$$

$$\ln(y) = x\ln(x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x\ln(x))$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(x)\cdot\ln(x) + x\cdot\frac{d}{dx}(\ln(x))$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\ln(x) + x\cdot\frac{1}{x}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln(x) + 1$$

$$\frac{dy}{dx} = y(\ln(x) + 1)$$

$$\frac{dy}{dx} = x^x(\ln(x) + 1)$$

So $f'(x) = x^x(\ln(x) + 1)$.

Find the derivatives of the following functions:

$$f(x) = (8x + 1)^{\sqrt{x}}$$

$$f(x) = \sec(x)^{\tan(x)+x^3}$$

$$f(x) = \left(\frac{1}{x}\right)^{\ln(3x)}$$