Topic 3: APPLICATIONS FOR NON SCIENCE

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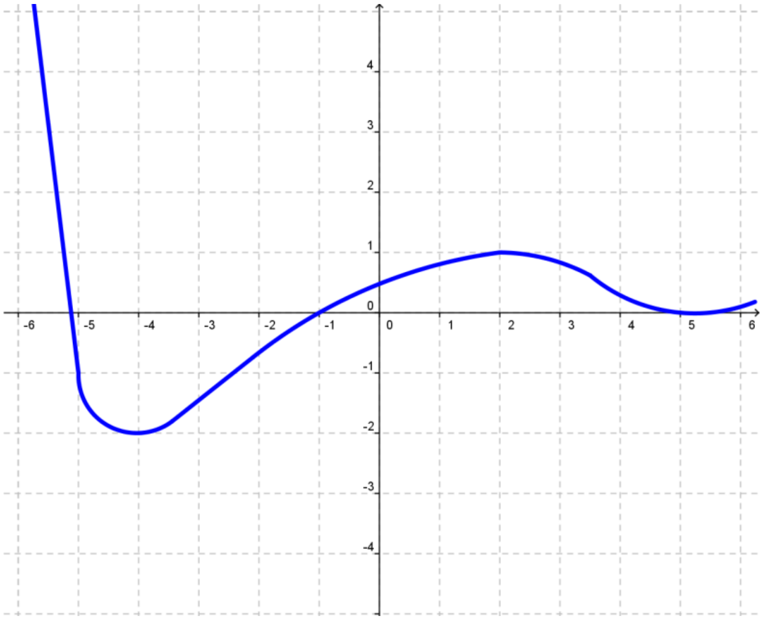
3.1: INCREASING & DECREASING FUNCTIONS

Observe the graph below and provide the information required:



Intervals over which the function is increasing:

Intervals over which the function is decreasing:

Intervals over which the slope of the tangent line is positive:

Intervals over which the slope of the tangent line is negative:

RECALL: A derivative measures how quickly a function $f(x)$ is changing with respect to x. Its value for a particular value of x indicates the rate of change of f compared to that of x. For example, if a function $f(x)$ is differentiable and $f'(1) = 2$, then a small variation of x near one will result in a variation roughly two times greater of $f(x)$. Put another way, we can say that, in the situation just described, the slope of the graph of $f(x)$ at $x = 1$ —and thus the slope of the tangent line of $f(x)$ at $x = 1$ — is 2.

So why will this help me sketch the graph of a function? Well, if we can categorize the different values of x according to the value of the derivative of $f(x)$, we can make the following conclusions:

$f'(x) > 0$	when	$f(x)$ is increasing
$f'(x) < 0$	when	$f(x)$ is decreasing
$f'(x) = 0$	when	$f(x)$ is level

Critical Values: Of course, in order for a function to change from being positive to negative (or vice versa), it must either pass over the x-axis or be discontinuous. So we can break a function that is not piecewise defined into intervals using the x-values where the function might change from positive to negative (or vice versa) by finding where it is zero or undefined. We call these x-values the **critical values** of the function. By using this method, we need only verify that a single point in an interval between critical values yields a positive value for the function in order to conclude that the function is positive over the entire interval.

Determine for when the functions below are increasing, decreasing, or level:

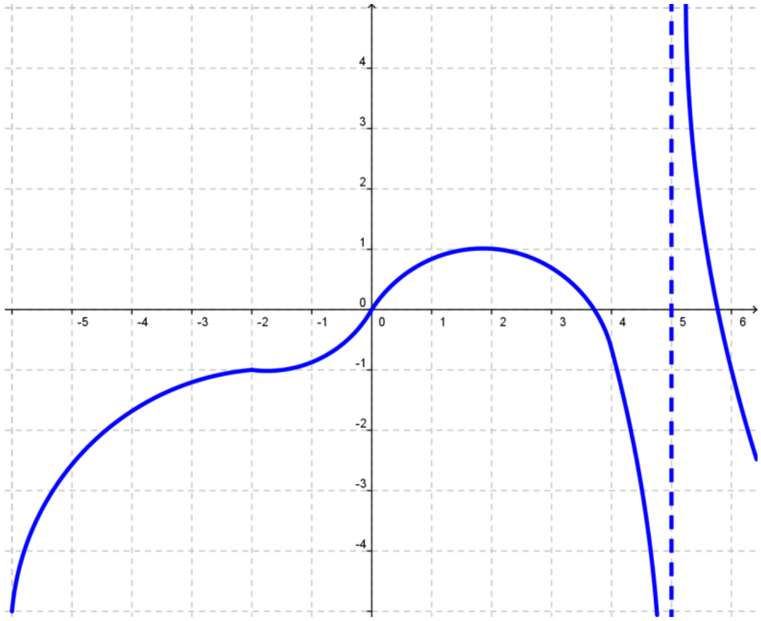
$$f(x) = e^{x^2+2x-24}$$

$$f(x) = \frac{-1}{(x+3)(2x-7)}$$

$$f(x) = x^5 + 9x^4$$

3.2: CONCAVITY

Observe the graph below and provide the information required:



Intervals over which the tangent line’s slope is increasing:

Intervals over which the tangent line’s slope is decreasing:

Intervals over which the curve is concave up:

Intervals over which the curve is concave down:

We calculate the second order derivative function $f''(x)$ by differentiating the first order derivative $f'(x)$. We can therefore conclude, based on the definition of a derivative, that the second order derivative of a function $f(x)$ measures the rate of change of $f(x)$’s rate of change or — should you prefer to think of it another way — it measures the rate of change of the slope of the graph of $f(x)$.

As a result, if the second order derivative of a function is positive, it indicates that the slope of the graph is increasing as x increases. In such cases, we say that the graph is **concave up** and it resembles a portion of an upward facing arc or a smile. Conversely, if the second derivative is negative for certain values of x, it indicates that the slope of the graph is decreasing. We say that in such a situation the graph is **concave down** and it resembles a portion of a downward facing arc or a frown.

If $f''(x)$ is zero, this indicates that the slope of the function is neither increasing nor decreasing. As a result, if $f''(x)$ is zero over an interval of values of x, this implies that the slope of the graph is constant over that interval and so $f(x)$ makes a straight line. However, the more common occurrence is that $f''(x) = 0$ only at a specific value of x. In these cases, it is possible to obtain what is called a **point of inflection**. These are points at which the graph of $f(x)$ switches from being concave up to being concave down and it can be recognized visually as an “S” (or backward “S”) shape in the curve.

$f''(x) > 0$	when	$f(x)$ is concave up
$f''(x) < 0$	when	$f(x)$ is concave down
$f''(x) = 0$	when	$f(x)$ has a point of inflection or makes a straight line

Determine when the following functions are concave up, concave down, or a point of inflection:

$f(x) = x^7 + 3x^6 + 10x - 1$

$f(x) = x^2 e^{3x}$

3.3: ASYMPTOTES

We must be aware of two kinds of asymptotes: vertical asymptotes and horizontal asymptotes.

Vertical asymptotes are commonly found by determining the x-values for which the function is undefined (due to division by zero). We may further confirm that there is, in fact, a vertical asymptote at $x = a$ if — in addition to the function being undefined at $x = a$ — the limits of the functions as x approaches a from both the positive and negative side tend to either infinity or negative infinity.

Horizontal asymptotes can be detected by taking the limits of the function as it approaches positive infinity and negative infinity. For example, if $\lim_{x \to \infty}\left(f(x)\right) = 16$, then we can conclude that the function will level off to a y-value that is increasingly close to 16 as the corresponding x-value gets larger and large (i.e. as x approaches infinity). In such a case, we would say that the function $f(x)$ has a horizontal asymptote at $y = 16$ on the right-hand side. We can similarly detect horizontal asymptotes on the left-hand side of the graph of $f(x)$ by evaluating the limit as x approaches negative infinity. If the limit at positive or negative infinity is ∞ or $-\infty$, then we can conclude that there is no horizontal tangent; the function's graph does not level off — instead it will consistently increase or decrease without bounds.

NOTE: It is possible to obtain only one horizontal asymptote (either on the left- or right-hand side of the graph of a function $f(x)$) or even to obtain two distinct horizontal asymptotes with different values.

Determine whether or not the following functions have horizontal and/or vertical asymptotes:

$$f(x) = \frac{2^x(x+8)^2}{5x^2 - 20}$$

$$f(x) = \sqrt{\frac{x^2+5}{3x^4}}$$

$$f(x) = \frac{3x(3x+2)(x-7)}{6x^2(6x-3)(9x+1)}$$

3.4: CURVE SKETCHING: A SUMMARY

A popular technique for graphing functions is to define $f(x)$, $f'(x)$, and $f''(x)$ by differentiating the given function. We then use algebra to find all of the **critical values** of these three functions. These are values for which the function is either zero or undefined and we call them critical because it is at these points that changes in the behaviour of the curve can take place, as $f(x)$, $f'(x)$, and $f''(x)$ have an opportunity to switch from positive to negative values, thus changing either the graph's position, increasing/decreasing property, or concavity. We use these critical values to dissect the number line into a series of critical values and the intervals which separate them. We then take note of the value of $f(x)$ at these critical values or by taking limits of the function as it approaches undefined points in order to have reference points for where the graph will pass and verify the sign (positive, negative, or zero) of the first and second derivatives at these point and at the intervals that separate them in order to obtain an idea of the curves behaviour for various values of x.

The entire procedure ends by taking the limit of the functions at positive and negative infinity in order to establish the graph's ultimate behaviour. For example, if $\lim_{x\to\infty}\left(f(x)\right) = \infty$, then we can conclude that the function will increase unboundedly as x increases; if $\lim_{x\to\infty}\left(f(x)\right) = -\infty$, then we know that the graph will decrease unboundedly as x increases; but if $\lim_{x\to\infty}\left(f(x)\right) = L$ for some real value L, then we can determine that the graph will ultimately approach a **horizontal asymptote** described by $y = L$.

For example, if we wanted to sketch the curve described by the function $f(x) = \frac{1}{3}x^{5/3}(8-x)$, we would be able to do so as follows:

Step 1: Find the first order derivative.

$$f'(x) = \frac{d}{dx}\left(\frac{1}{3}x^{5/3}(8-x)\right) = \frac{8}{9}x^{2/3}(5-x)$$

Step 2: Find the second order derivative.

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{8}{9}x^{2/3}(5-x)\right) = \frac{40}{27x^{1/3}}(2-x)$$

Step 3: Find the critical values (where $f(x)$, $f'(x)$, or $f''(x)$ are zero or undefined).

$f(x) = \frac{1}{3}x^{5/3}(8 - x)$ is zero when $x = 0$ or 8 and is never undefined.

$f'(x) = \frac{8}{9}x^{2/3}(5 - x)$ is zero when $x = 0$ or 5 and is never undefined.

$f''(x) = \frac{40}{27x^{1/3}}(2 - x)$ is zero when $x = 2$ and is undefined when $x = 0$.

Step 4: Build a table based on the critical values and limits and interpret the results.

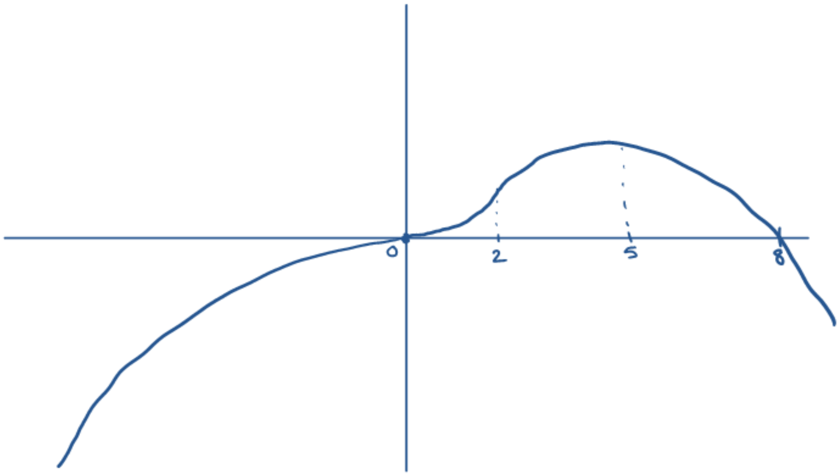
x	$-\infty$	$(-\infty, 0)$	0	$(0,2)$	2	$(2,5)$	5	$(5,8)$	8	$(8, \infty)$	∞
$f(x)$	$-\infty$	$-$	0	$+$	$+$	$+$	$+$	$+$	0	$-$	$-\infty$
$f'(x)$	$+$	$+$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$
$f''(x)$	$-$	$-$	Undef.	$+$	0	$-$	$-$	$-$	$-$	$-$	$-$

The second line tells us that the function is above the x-axis on the interval $(0,8)$, on the x-axis at $x = 0$ and $x = 8$, and below the x-axis on the intervals $(-\infty, 0)$ and $(8, \infty)$. The limits at $\pm\infty$ show that there is no horizontal tangent.

The third line tells us that the function is increasing on the intervals $(-\infty, 0)$ and $(0,5)$, it is level at $x = 0$ and $x = 5$, and it is decreasing on the interval $(5, \infty)$. Base on this, we can state that $x = 5$ is a relative maximum for the function.

The fourth line tells us that the function is concave up on the interval $(0,2)$, it is concave down on the intervals $(-\infty, 0)$ and $(2, \infty)$, and there is a point of inflection at $x = 2$.

Step 5: Sketch the curve.



For future reference, let's establish the shapes of some different types of curves:

Increasing	Decreasing	Increasing	Decreasing
Concave up	Concave up	Concave down	Concave down

Sketch the curves of the following functions:

$$f(x) = \frac{x}{x^2 - 9}$$

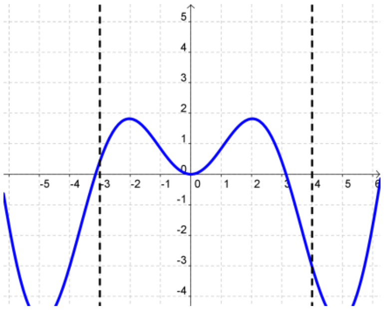
$$f(x) = x + 3x^{2/3}$$

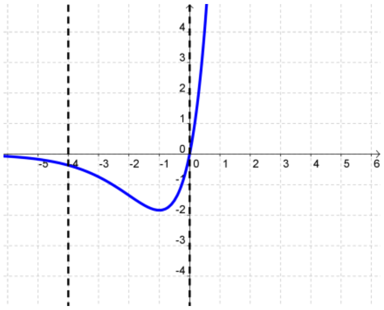
$$f(x) = x^{\frac{1}{3}}(x - 1)$$

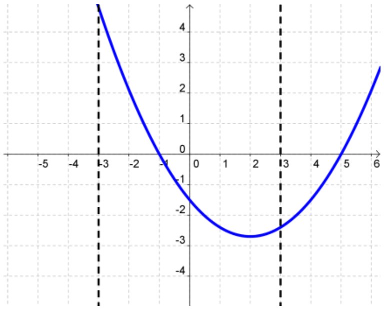
$$f(x) = \frac{6 - 6x}{x^2}$$

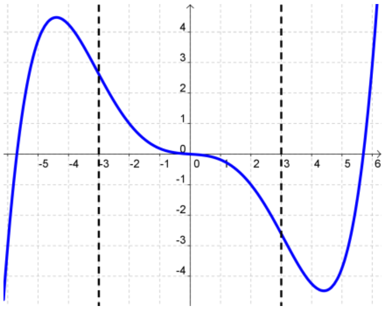
3.5: OPTIMIZATION

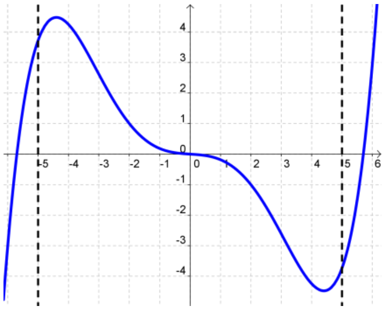
Identify the maximum and minimum values (we call these absolute extrema) of the functions described by the curves as long as the x-value is located in between the two dotted lines.

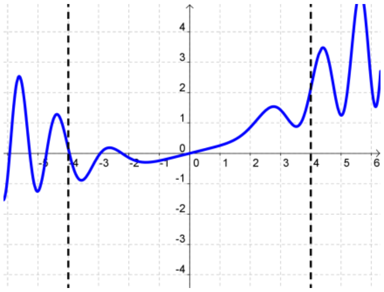












We can conclude that the largest and smallest values that a function can take will either be

(1) At a relative maximum or minimum (i.e. when $f'(x) = 0$)

OR

(2) At the ends of an interval over which we are restricting x-values.

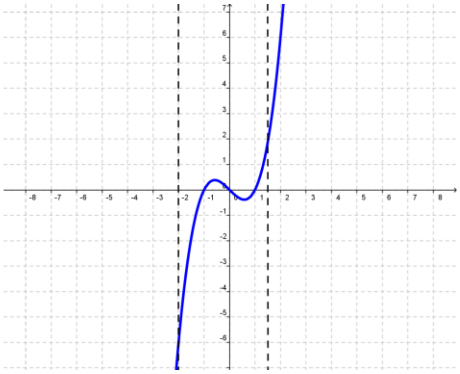
As a result, when seeking the absolute extrema of a function $f(x)$, we must verify the value of $f(x)$ whenever $f'(x) = 0$ and at the ends of any interval that might restrict the value of x.

For example, if we were asked to find the maximum and minimum values of the function $f(x) = x^3 - x$ on the interval $[-2,1.5]$, we would naturally seek to find the relative maximums and minimums of $f(x)$ by determining for which values of x we obtain $f'(x) = 0$ as follows:

$$f'(x) = \frac{d}{dx}(x^3 - x) = 3x^2 - 1$$

$$f'(x) = 0 \Rightarrow 3x^2 - 1 = 0 \Rightarrow \pm\sqrt{\frac{1}{3}}$$

As a result, we would consider that the absolute maximum and minimum might occur at $x = \pm\sqrt{\frac{1}{3}}$. Therefore, let's see what the value of the function would be at these values:



$$f\left(-\sqrt{\frac{1}{3}}\right) = \left(-\sqrt{\frac{1}{3}}\right)^3 - \left(-\sqrt{\frac{1}{3}}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} \approx 0.3849.$$

$$f\left(\sqrt{\frac{1}{3}}\right) = \left(\sqrt{\frac{1}{3}}\right)^3 - \sqrt{\frac{1}{3}} = -\frac{2}{3}\sqrt{\frac{1}{3}} \approx -0.3849.$$

Some might be tempted to stop here and to simply claim that the absolute maximum of the function is $f\left(-\sqrt{\frac{1}{3}}\right) = \frac{2}{3}\sqrt{\frac{1}{3}}$ and that the absolute minimum is $f\left(\sqrt{\frac{1}{3}}\right) = -\frac{2}{3}\sqrt{\frac{1}{3}}$, but this would be incorrect!

We must also verify the values of the function at the ends of the interval $[-2, 1.5]$:

$$f(-2) = (-2)^3 - (-2) = -6$$

$$f(1.5) = 1.5^3 - 1.5 = 1.875$$

So the true absolute maximum of $f(x)$ on the interval $[-2,1.5]$ is 1.875 (when $x = 1.5$) and the absolute minimum of $f(x)$ on this interval is -6 (when $x = -2$). This is easily confirmed by consulting the graph of $f(x) = x^3 - x$.

As a result, we should establish the following procedure for finding the absolute maximum and absolute minimum values of a function over an interval $[a, b]$:

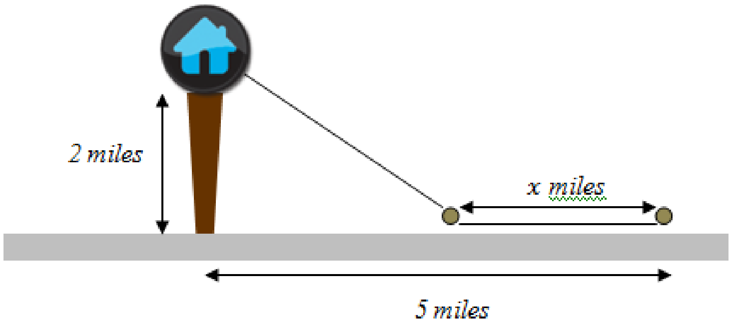
(1) Find a function $f(x)$ to represent the value that you are seeking to optimize.
(2) Find the derivative of the function $f'(x)$ and any values of x which make $f'(x) = 0$. (Let's call these values $x_1, x_2, x_3, \ldots$)
(3) Calculate $f(x_1)$, $f(x_2), f(x_3), \ldots$ as well as the values of $f(x)$ at the ends of the interval being considered, i.e. $f(a)$ and $f(b)$. (Note that if there are no restrictions on the values that x can take, we need not determine $f(a)$ and $f(b)$.)
(4) Choose the largest of the values calculated in step 2 to be the absolute maximum of $f(x)$ and the smallest of the values to be the minimum of $f(x)$ on the interval $[a, b]$.

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that will minimize the cost of material in their construction.

Find the point on the line $3x + y = 0$ that is closest to the point $(0, -3)$.

MetroMedia is asked to provide service to a customer whose house is located 2 miles from the road along which the cable is buried. The nearest connection box for the cable is located 5 miles down the road.



If the installation cost is \$500 per mile along the road and \$700 per mile off the road, express the total cost C of installation as a function of the distance x (in miles) from the connection box to the point where the cable installation turns off the road. Assuming that the cable installation will turn off the road somewhere in between the connection box and he intersection of the house's entrance and the road, what distance x would prove to be most economical? Which distance would be the most costly?

What are the absolute maximum and absolute minimum values of the function $f(x) = e^{x^2-2x}$ on the interval $[-3,6]$?

When the values of x are not restricted to a specific interval, we can use limits to find the values of the function at $x \to \pm\infty$. However, it may be easier to simply check if the function is increasing or decreasing before and after the value of x that yields $f'(x) = 0$ in order to verify whether it is a relative maximum (increasing before and decreasing after), a relative minimum (decreasing before and increasing after), or a point of inflection (either increasing or decreasing both before AND after).

Suppose that $f'(a) = 0$:

$f'(x)$ **at a point** $x < a$	$f'(x)$ **at a point** $x > a$	**Sketch**	**Conclusion about what happens at** $x = a$
$+$	$-$		
$-$	$+$		
$-$	$-$		
$+$	$+$		

Find all of the relative extrema of the function $f(x) = 2x^3 + \frac{1}{2}x^2 - 2x + 10$. Specify whether they are relative maximums or relative minimums using the first derivative test.

Second Derivative Test

Another option would be to examine the value of the second order derivative at values of x that cause the first derivative to be zero. If the function is clearly concave up ($f''(x)$ is positive), then we must conclude that we have found a relative minimum. If the function is concave down ($f''(x)$ is negative), then we have discovered a relative maximum.

Suppose that $f'(a) = 0$:

$f''(a)$	**Sketch**	**Conclusion about what happens at $x = a$**
$+$		
$-$		

NOTE: If the value of $f''(a) = 0$, a multitude of options are available to us. Often, this is an indication that we have uncovered a point of inflection at $x = a$, but this is not guaranteed; the curve may simply make a straight line at that point.

Find and categorize the relative maximums, relative minimums, and possible points of inflection of the following functions using the second derivative test:

$f(x) = x^7 + 3x^6 + 10x - 1$

$f(x) = x^2 e^{3x}$

3.6: RATES OF CHANGE: MARGINAL FUNCTIONS

In commerce, we can often predict the demand that the consumer public will have for an item according to the price.

Pricing function $p(x)$: A function which indicates the price at which one unit should be sold in order to sell x units.

Revenue function $R(x)$: Revenue is a term used to explain the amount of money flowing into the company (usually the total of all sales). Consequently, in order to find the total revenue of a production venture, one needs only multiply the number of units being purchased by the price at which they are being sold. In other words, if we are to let the function $R(x)$ represent the total revenue of production, we have that

$$R(x) = \underbrace{x}_{\textit{number of units being sold}} \cdot \underbrace{p(x)}_{\textit{selling price per unit}}.$$

M**arginal revenue $R'(x)$**: This term refers to the approximate increase in total revenue which would occur by increasing production by one unit. As a result, the marginal revenue corresponds to the rate of change of the total revenue with respect to an increase in the number of units sold: $R'(x)$.

The pricing function of an MP3 player is $p(x) = -0.004x + 100$. Identify the total revenue function $R(x)$ and determine the marginal revenue when $x = 750$. What does this mean?

The pricing function of a new car by Honda is $p(x) = -\frac{60}{\sqrt{x}} + 30000$. Identify the total revenue function $R(x)$ and determine the marginal revenue when $x = 10000$.

The “marginal” concept can also be extended to a number of values in business. For example, the marginal profit and marginal cost would represent the increase in profit and cost, respectively, caused by increasing production by one unit. We would be able to approximate these values quite nicely by evaluating the derivatives of the profit function and cost function, respectively, at the appropriate level of production.

Profit $P_R(x)$: This indicates the amount of money left over for the company once the costs of operation (materials, wages, rent, power,...) have been taken into account:

$$P_R(x) = R(x) - C(x)$$

Marginal Profit $P'_R(x)$: The approximate increase in profit that would result from increasing production by one unit.

If you have a function describing the average cost of production for x units $\bar{C}(x) = \frac{3000}{x} + 200$, how would you find a function to describe the overall cost of production $C(x)$?

A pool table manufacturer discovers that the pricing function for their tables is of the form $p(x) = 10000 - 50x$. The average cost of production of a pool table is defined by the function $\bar{C}(x) = \frac{3000}{x} + 200$.Find the revenue function $R(x)$, the cost function $C(x)$, and the profit function $P_R(x)$, as well as the marginal revenue, cost, and profit at a production level of $x = 200$.

3.7: BUSINESS AND ECONOMICS APPLICATIONS

Of course, with optimization comes its application to the business world. Let's start with a few obvious examples:

Determine the production level that will maximize the profit for a company with cost function $C(x) = 84 + 1.26x - 0.01x^2 + 0.00007x^3$ and the demand function illustrating the price that will allow for the sale of x units is $p(x) = 3.5 - 0.01x$.

A company has determined that the pricing function for one of its products is

$$p(x) = -x^2 + 450x + 52{,}500$$

where x represents the number of units produced (and sold). What production level will yield a maximum revenue?

A store has been selling 200 compact disc players a week at \$350 each. A market survey indicates that for each \$10 reduction in sale price, the number of sets sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

A company estimates that the cost (in dollars) of producing x units of a product can be modeled by $C(x) = 800 + 0.04x + 0.0002x^2$. Find the production level that minimizes the average cost per unit.

The marketing department of a business has determined that the demand for a product can be modeled by $p(x) = \frac{50}{\sqrt{x}}$. The cost of producing x units is given by $C(x) = 0.5x + 500$. What price will yield a maximum profit?

3.8: ELASTICITY

Price elasticity of demand: We say that a product has a price that is **elastic** if reducing the price yields large changes in the number of units sold. Conversely, we say that a price is **inelastic** if reducing the price does not yield much change in the product's sales. The way that we determine what kind of change qualifies as large enough to merit the term "elastic" is by comparing the percent change in quantity sold to the percent change in price as follows:

$$Price\ elasticity\ of\ demand = \eta = \frac{rate\ of\ change\ in\ demand}{rate\ of\ change\ in\ price}$$

$$= \frac{\Delta x/x}{\Delta p/p}$$

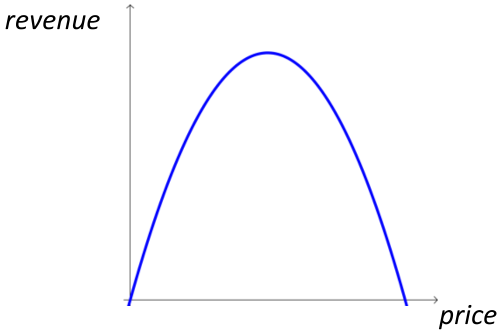
$$= \frac{p/x}{\Delta p/\Delta x}$$

$$= \frac{p/x}{dp/dx}$$

$$= \frac{p(x)}{x \cdot p'(x)}$$

Why do we care?

The value	Elasticity	What it means...
$\lvert\eta\rvert > 1$	elastic	slight increase in price → decrease in revenue slight decrease in price → increase in revenue
$\lvert\eta\rvert < 1$	inelastic	slight increase in price → increase in revenue slight decrease in price → decrease in revenue
$\lvert\eta\rvert = 1$	unit elastice	slight increase in price → revenue does not change slight decrease in price → revenue does not change



The demand function (i.e. pricing function) for a product is $p(x) = 21 - 1.5\sqrt{x}$ if $0 \leq x \leq 196$. Find the intervals on which the elastic, inelastic, and of unit elasticity. Use this result to describe the behaviour of the revenue function.

Find the intervals on which the demand function $p(x) = 34 - 2\sqrt{x}$ (for $0 \leq x \leq 289$) is elastic, inelastic, and of unit elasticity.