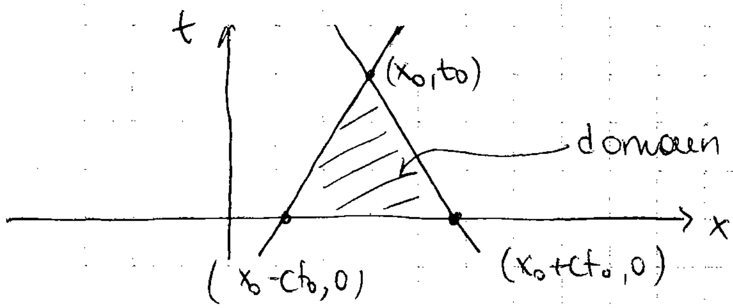
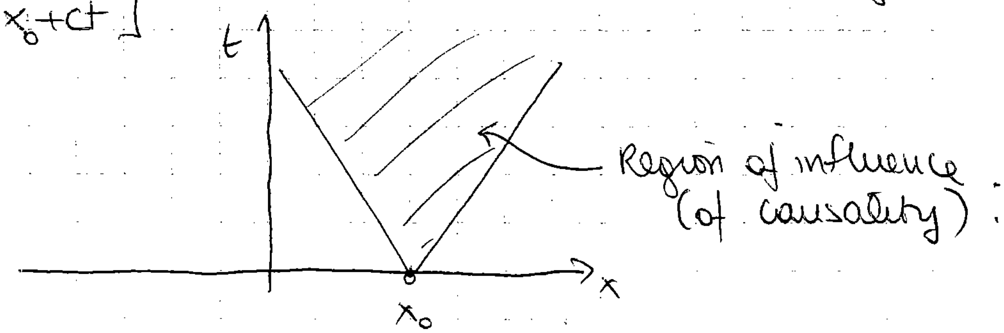
5.3 Interpretation of the characteristics

- Observing the functional form of the solution $u(x,t)$, we see that the value of the solution at a point (x_0, t_0) depends only on the initial conditions within the interval $[x_0 - ct_0, x_0 + ct_0]$.



- Conversely, any point (x_0) in the initial data influences the solution at a later time within a region $[x_0 - ct, x_0 + ct]$



The characteristics delimit both domain of dependence & region of influence. Information is transported along/within characteristics in a causal way

Note: if $g(x) = 0 \ \forall x$ then the characteristics transport all of the information and the solution is an average of the information from $x_0 - ct_0$ and $x_0 + ct_0$.

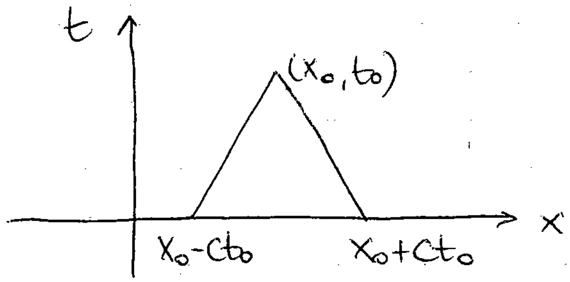
5.4 Non Homogeneous wave equation

(Initial value problem on an ∞ domain)

Suppose we want to solve
$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

This corresponds, for instance, to the oscillations of an ∞ string under external forcing.

Let's consider the triangle in the (x,t) plane



and integrate the PDE over the triangle:

$$\iint_{\Delta} (u_{tt} - c^2 u_{xx})\,dx\,dt = \iint_{\Delta} F(x,t)\,dx\,dt$$

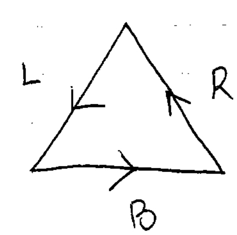
Let $P = u_t$ and $Q = c^2 u_x$ then

$$\iint_{\Delta} (P_t - Q_x)\,dx\,dt = \iint_{\Delta} F(x,t)\,dx\,dt$$

$$= -\oint_{\Gamma} P(x,t)\,dx + Q(x,t)\,dt$$ (following Green's formula).

Γ ← the contour of the triangle.

Let's define Γ as $B + L + R$
(↑ base, ↑ left, ↑ right)



Parametric representations of the contour Γ

On the base: $dt = 0$, $t=0$

so $$\int_B P(x,t)dx + Q(x,t)dt = \int_B P(x,0)dx$$

$$= \int_B u_t(x,0)dx = \int_{x_0-ct_0}^{x_0+ct_0} g(x)dx$$

On the left side $x-ct = \text{constant} = x_0-ct_0$

so $dx - c\,dt = 0$

so $$\int_L P(x,t)dx + Q(x,t)dt = \int_L u_t(x,t)dx + c^2 u_x(x,t)dt$$

$$= \int_L u_t(x,t)\,c\,dt + c^2 u_x(x,t)\frac{dx}{c}$$

not surprising: L is a characteristic

$$= c\int_L du = c\left[-u(x_0,t_0) + u(x_0-ct_0,0)\right]$$

$$= c\left[u(x_0,t_0) + f(x_0-ct_0)\right]$$

Similarly on the right side

$$\int_R P\,dx + Q\,dt = -c\left[u(x_0,t_0) - u(x_0+ct_0,0)\right]$$

$$= -c\left[u(x_0,t_0) - f(x_0+ct_0)\right]$$

So, putting it all together we get

$$-2c\,u(x_0,t_0) + c\left(f(x_0-ct_0) + f(x_0+ct_0)\right) + \int_{x_0-ct_0}^{x_0+ct_0} g(x)dx$$

$$= -\iint_\Delta F(x,t)\,dx\,dt$$

$$\Rightarrow u(x_0,t_0) = \frac{1}{2}\left[f(x_0-ct_0)+f(x_0+ct_0)\right] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(x)dx + \frac{1}{2c}\iint_{\Delta(x_0,t_0)} F(x,t)\,dx\,dt$$

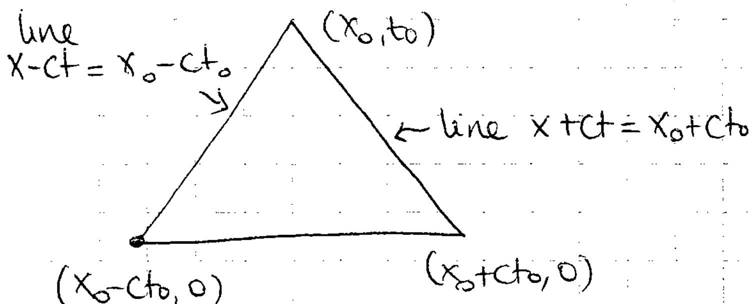
This is also known as d'Alembert's formula.

<u>Example</u>: $\begin{cases} u_{tt} - 9u_{xx} = 2\sinh(x) \\ u(x,0) = x \\ u_t(x,0) = \sin(x) \end{cases}$

Here, we use the formula with
$F(x,t) = 2\sinh(x)$, $c = 3$, $f(x) = x$, $g(x) = \sin x$ so

$$u(x_0,t_0) = \frac{1}{2}\left[(x_0 - 3t_0) + (x_0 + 3t_0)\right] + \frac{1}{6}\int_{x_0-3t_0}^{x_0+3t_0} \sin(x)\,dx - \frac{1}{6}\iint_\Delta F(x,t)\,dx\,dt$$

To perform the double integral, we need to find the bounds.



There are two ways of doing integral: first integrate x then t, or vice versa.

Way # ①:
$$\int_{t=0}^{t=t_0} \int_{x = ct + x_0 - ct_0}^{x = -ct + x_0 + ct_0} F(x,t)\,dx\,dt$$

Way # ②:
$$\int_{x = x_0 - ct_0}^{x = x_0} \int_{t=0}^{t = \frac{x - x_0 + ct_0}{c}} F(x,t)\,dx\,dt + \int_{x = x_0}^{x = x_0 + ct_0} \int_{t=0}^{t = \frac{-x + x_0 + ct_0}{c}} F(x,t)\,dx\,dt.$$

Here let's choose #1:
$$\iint_\Delta F = \int_{t=0}^{t=t_0} 2\left[\cosh(-ct + x_0 + ct_0) - \cosh(ct + x_0 - ct_0)\right] dt \quad \text{with } c = 3.$$
$$= -\frac{2}{3}\sinh(x_0) - \frac{2}{3}\sinh(x_0) + \frac{2}{3}\sinh(x_0 + 3t_0) + \frac{2}{3}\sinh(x_0 - 3t_0)$$

$$\Rightarrow u(x_0,t_0) = x_0 + \frac{1}{6}\left(-\cos(x_0 + 3t_0) + \cos(x_0 - 3t_0)\right) - \frac{2}{9}\sinh(x_0)$$
$$+ \frac{1}{9}\sinh(x_0 + 3t_0) + \frac{1}{9}\sinh(x_0 - 3t_0) = x_0 + \frac{1}{3}\sin(x_0)\sin(3t_0)$$
$$- \frac{2}{9}\sinh(x_0) + \frac{2}{9}\sinh(x_0)\cosh(3t_0).$$

5.5 The Energy Method : Existence & uniqueness of solutions

5.5.1 Equivalent condition to uniqueness

Consider two associated problems

(P1) :

$$u_{tt} = c^2 u_{xx}$$

$u(x,0) = f(x)$ $\quad$ $u(a,t) = h(t)$

$u_t(x,0) = g(x)$ $\quad$ $u(b,t) = H(t)$

(P2)

$$u_{tt} = c^2 u_{xx}$$

$u(x,0) = 0$ $\quad$ $u(a,t) = 0$

$u_t(x,0) = 0$ $\quad$ $u(b,t) = 0$

P2 is said to have homogeneous ICs and BCs.

Lemma : The solution to P1 is unique
$\iff$ The only solution to P2 is $u \equiv 0$.

Proof $\Leftarrow$ (by negation).

Suppose we know two solutions to P_1; namely $u_1(x,t)$ and $u_2(x,t)$ then

$V(x,t) = u_1(x,t) - u_2(x,t)$ is a non-zero solution of P_2. $\square$

The proof of $\Rightarrow$ is similar.

Therefore, to prove that the solution to a given linear PDE is unique, it is enough to show that the solution to the associated problem with homogeneous BCs and ICs is necessarily $u \equiv 0$.
The Energy method is an easy way of doing just that.

5.5 2 Example of the Energy Method

Given the PDE $u_{tt} = c^2 u_{xx}$, we now define the function

$$E(t) = \int_a^b \left[u_t^2 + c^2 u_x^2 \right] dx$$

where a and b are the boundaries of the domain considered.

Then ① $E(t) \geq 0$ (by construction) and

② $$\frac{dE}{dt} = \int_a^b \frac{\partial}{\partial t}(u_t^2) + c^2 \frac{\partial}{\partial t}(u_x^2)\, dx$$

$$= \int_a^b 2u_t u_{tt} + 2c^2 u_{xt} u_x \, dx$$

$$= \int_a^b \left[2u_t u_{tt} + 2c^2 \frac{\partial}{\partial x}(u_x u_t) - 2c^2 u_t u_{xx} \right] dx$$

$$= \int_a^b 2c^2 \frac{\partial}{\partial x}(u_x u_t)\, dx$$

$$= 2c^2 \left[u_x(b,t) u_t(b,t) - u_x(a,t) u_t(a,t) \right]$$

If the BCs are such that either u_x or $u_t = 0$ on the boundaries then $\frac{dE}{dt} = 0$ at all times t.

Now, at $t=0$, the "Energy" E of problem P_2 is 0. Since $\frac{dE}{dt} = 0$, then $E = 0$ for all times

$\Rightarrow$ for all t, $\int_a^b u_t^2 + c^2 u_x^2 \, dx = 0$

$\Rightarrow$ $u_t = u_x = 0$ for all t and x

$\Rightarrow$ $u = 0$ for all x and t.

Note: This general concept can also be applied to other PDEs. The idea is derived from the physical concept of energy, but for most PDEs $E(t)$ only needs to satisfy

$$\begin{cases} E(t) \geq 0 \\ \dfrac{dE}{dt} \leq 0 \\ E(t) \text{ depends on } u(x,t) \text{ and/or its derivatives.} \end{cases}$$

For the heat equation, for example,

$$\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \end{cases}$$

$\rightarrow$ use $E(t) = \frac{1}{2} \int_0^L u^2(x,t)\, dx$.

$\rightarrow$ homework: Prove that it has the required properties