MTH141 Practice Questions for Test 1

1.2 B Find a unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = (3,4)$. Find a unit vector $\boldsymbol{U}$ perpendicular to $\boldsymbol{u}$. How many possibilities for $\boldsymbol{U}$?

Solution For a unit vector $\boldsymbol{u}$, divide $\boldsymbol{v}$ by its length $\|\boldsymbol{v}\| = 5$. For a perpendicular vector $\boldsymbol{V}$ we can choose $(-4,3)$ since the dot product $\boldsymbol{v}\cdot\boldsymbol{V}$ is $(3)(-4)+(4)(3)=0$. For a *unit* vector $\boldsymbol{U}$, divide $\boldsymbol{V}$ by its length $\|\boldsymbol{V}\|$:

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{(3,4)}{5} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad \boldsymbol{U} = \frac{\boldsymbol{V}}{\|\boldsymbol{V}\|} = \frac{(-4,3)}{5} = \left(-\frac{4}{5}, \frac{3}{5}\right)$$

The only other perpendicular unit vector would be $-\boldsymbol{U} = (\frac{4}{5}, -\frac{3}{5})$.

Solve the following system

$$\begin{aligned} x + 3y + 2z &= -3 \\ 2x + 2y + 2z &= -2 \\ 3x + 5y + 4z &= -5 \end{aligned}$$

$x = 1, y = 1, z = 0$. The solution becomes $\boldsymbol{x} = (1,1,0)$.

1. [(6+5) marks] Let $\mathbf{p} = (k,1)$ and $\mathbf{q} = (1,2)$.

(a) Find k such that the angle between $\mathbf{p}$ and $\mathbf{q}$ is $\frac{\pi}{4}$.

$$\measuredangle(p,q) = \frac{\pi}{4} \Rightarrow \cos\frac{\pi}{4} = \frac{p\cdot q}{\|p\|\,\|q\|} = \frac{k+2}{\sqrt{k^2+1^2}\cdot\sqrt{1^2+2^2}} \Rightarrow$$

$$\frac{1}{\sqrt{2}} = \frac{k+2}{\sqrt{k^2+1}\cdot\sqrt{5}} \quad /^2 \Rightarrow \frac{1}{2} = \frac{(k+2)^2}{(k^2+1)\cdot 5} \Rightarrow 5(k^2+1) = 2(k^2+4k+4)$$

$$\Rightarrow 5k^2+5 = 2k^2+8k+8 \Rightarrow 3k^2-8k-3=0$$

$$\Rightarrow k_{1,2} = \frac{8\pm\sqrt{64+36}}{6} = \frac{8\pm10}{6} \Rightarrow k_1 = 3, \quad k_2 = -\frac{1}{3}$$

(b) For $k = 1$ find $\text{proj}_{\mathbf{q}}\mathbf{p}$.

$$\text{Proj}_q P = \frac{P\cdot q}{\|q\|^2}\cdot q = \frac{1+2}{1^2+2^2}(1,2) = \frac{3}{5}(1,2) = \left(\frac{3}{5}, \frac{6}{5}\right)$$

2. (a) [6 marks] Find an equation of the plane whose points are equidistant from $(-1,0,1)$ and $(0,-1,-1)$.

Let $(x,y,z)\in\mathbb{R}^3$. We want $d\big((x,y,z),(-1,0,1)\big) = d\big((x,y,z),(0,-1,-1)\big)$

$$\Rightarrow \sqrt{(x+1)^2+y^2+(z-1)^2} = \sqrt{x^2+(y+1)^2+(z+1)^2}$$

$$\Rightarrow \cancel{x^2}+2x+\cancel{1}+\cancel{y^2}+\cancel{z^2}-2z+\cancel{1} = \cancel{x^2}+\cancel{y^2}+2y+\cancel{1}+\cancel{z^2}+2z+\cancel{1}$$

$$\Rightarrow 2x-2y-4z=0$$

$$\Rightarrow x-y-2z=0$$

(b) [6 marks] Show that the line $x = 0,\ y = t,\ z = t$ intersects the plane $x + 2y + 2z = 2$ and find the point of intersection.

Plug the equations $x=0, y=t, z=t$ into $x+2y+2z=2$.

$0 + 2t + 2t = 2 \Rightarrow 4t = 2 \Rightarrow t = \frac{1}{2}$

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The intersecting point is $x=0$, $y=\frac{1}{2}$, $z=\frac{1}{2}$, i.e. $(0, \frac{1}{2}, \frac{1}{2})$.

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3. [(10+5) marks] Consider the linear system

$$\begin{aligned} x_1 + x_2 - 2x_3 + 4x_4 &= a + 1 \\ 2x_1 + 2x_2 - 3x_3 + x_4 &= b + 2 \\ 3x_1 + 3x_2 - 4x_3 - 2x_4 &= c + 3. \end{aligned}$$

(a) What conditions must exist between a, b, and c for the above linear system to be consistent?

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & a+1 \\ 2 & 2 & -3 & 1 & b+2 \\ 3 & 3 & -4 & -2 & c+3 \end{array}\right] \xrightarrow[(-3)R_1+R_3]{(-2)R_1+R_2} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & a+1 \\ 0 & 0 & 1 & -7 & b-2a \\ 0 & 0 & 2 & -14 & c-3a \end{array}\right] \xrightarrow{(-2)R_2+R_3}$$

1 4 2

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & a+1 \\ 0 & 0 & 1 & -7 & b-2a \\ 0 & 0 & 0 & 0 & a+c-2b \end{array}\right]$$

The system is going to be consistent iff $a+c-2b=0$ iff $2b = a+c$

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(b) Solve the above system taking $a = b = c = 1$.

Putting $a=b=c=1$ in the last matrix above we get

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$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 2 \\ 0 & 0 & 1 & -7 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{2R_2+R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -10 & 0 \\ 0 & 0 & 1 & -7 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$ the RREF of the augmented matrix of the system when $a=b=c=1$

The corresponding system is

$$\begin{aligned} x_1 + x_2 - 10x_4 &= 0 \\ x_3 - 7x_4 &= -1 \end{aligned}$$

Solving for the leading variables x_1, x_3 in terms of free variables x_2, x_4 we get

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$$\begin{aligned} x_1 &= 10t - s \\ x_2 &= s \\ x_3 &= -1 + 7t \\ x_4 &= t \end{aligned}$$ where $s, t \in \mathbb{R}$

4. [(3+1+4+4) marks] **Note:** Answers to parts (c) and (d) depend on (a) !

(a) Solve the equation $z^2 - 2z + 4 = 0$ by the quadratic formula.

$$z_{1,2} = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2i\sqrt{3}}{2}$$

$$z_1 = 1 + i\sqrt{3}$$

$$z_2 = 1 - i\sqrt{3}$$

(b) Check your result by substituing the solutions into the equation.

$$(1+i\sqrt{3})^2 - 2(1+i\sqrt{3}) + 4 = 1 + \cancel{2i\sqrt{3}} - 3 - 2 - \cancel{2i\sqrt{3}} + 4 = 0 \quad 0.5$$

$$(1-i\sqrt{3})^2 - 2(1-i\sqrt{3}) + 4 = 1 - \cancel{2i\sqrt{3}} - 3 - 2 + \cancel{2i\sqrt{3}} + 4 = 0 \quad 0.5$$

(c) Express the solutions z_1 and z_2 of the equation from (a) in the polar form.

$$|z_i| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2 \quad i = 1,2. \qquad \cos\theta_i = \frac{1}{2} \quad \sin\theta_1 = \frac{\sqrt{3}}{2} \quad \sin\theta_2 = -\frac{\sqrt{3}}{2}$$

$$\arg(z_1) = \frac{\pi}{3} \qquad \arg(z_2) = -\frac{\pi}{3}$$

$$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z_2 = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

(d) Express $z_1^{2010} + z_2^{2010}$ as a real number, where z_1 and z_2 are the solutions of the equation from (a). Your answer should not contain any trigonometric terms.

$$z_1^{2010} + z_2^{2010} = 2^{2010}\left(\cos\frac{2010\pi}{3} + i\sin\frac{2010\pi}{3}\right) + 2^{2010}\left(\cos\frac{2010\pi}{3} - i\sin\frac{2010\pi}{3}\right)$$

$$= 2^{2011} \underbrace{\cos 670\pi}$$

$$= 2^{2011}$$