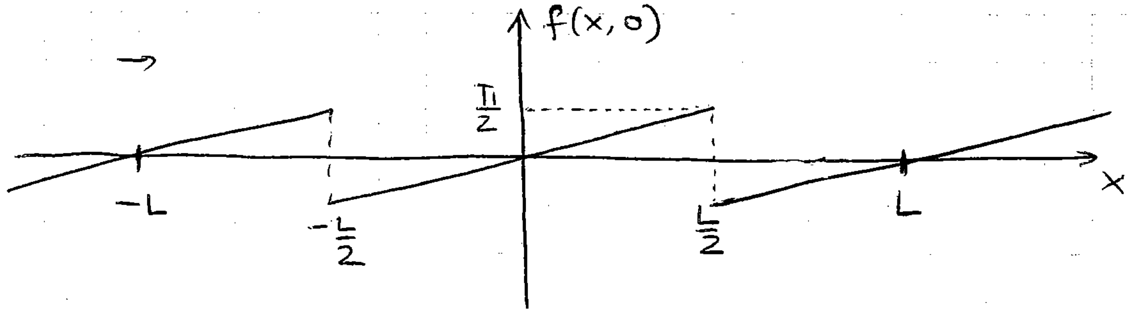
So the solution for $T(x,t)$ is

$$T(x,t) = T_1 - \frac{T_1}{L}x + \sum{}' b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 Dt}{L^2}}$$

Where $$\sum_{n=1}^{\infty}{}' b_n \sin\left(\frac{n\pi x}{L}\right) = f(x,0)$$

$\Rightarrow$ Construct a Fourier Series for $f(x,0)$ by assuming
- it is periodic with period $2L$
- it is antisymmetric



So $$b_n = \frac{1}{L}\int_{-L}^{L} f(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L f(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^{\frac{L}{2}} \frac{T_1 x}{L}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$+ \frac{2}{L}\int_{\frac{L}{2}}^{L} \frac{T_1}{L}(x-L)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2T_1}{L^2}\left[x\left(\frac{-L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{2T_1}{L^2}\left(+\frac{L}{n\pi}\right)\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$

$$- \frac{2T_1}{L}\left[\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_{\frac{L}{2}}^{L}$$

$$= \cancel{-\frac{2T_1}{n\pi}\cos(n\pi)} + \cancel{\frac{2T_1}{n\pi}\cos n\pi} - \frac{2T_1}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

so $b_n = -\dfrac{2T_1}{n\pi}\cos\left(\dfrac{n\pi}{2}\right)$

$$\Rightarrow T(x,t) = T_1\left[1 - \frac{x}{L} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2 Dt}{L^2}}\right]$$

Note :

Each mode decays with a different typical timescale: the decay time for mode n is

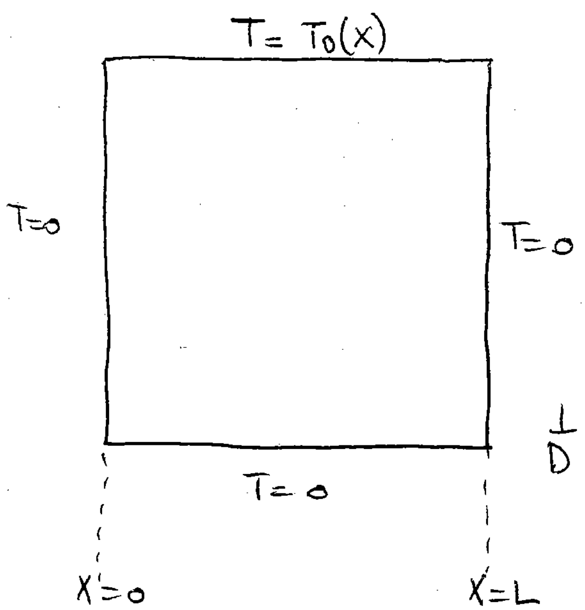
$$\tau_n = \frac{L^2}{\pi^2 n^2 D}$$

→ the higher the degree n, the faster the decay

→ diffusion smoothes out small scale faster than large scales

4.3 Laplace Equation

Consider a square plate with sides held at the following temperatures:



What is the steady-state temperature profile on the plate as a result of this heating?

→ Solve

$$\frac{1}{D}\frac{\partial T}{\partial t} = \boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

Separation of variables

$$T(x,y) = A(x)\,B(y)$$

$$\Rightarrow \quad \frac{1}{A}\frac{d^2A}{dx^2} = K$$

$$\frac{1}{B}\frac{d^2B}{dy^2} = -K$$

Note that

- if $K>0$ then $\begin{cases} A \text{ has exponential behaviour} \\ B \text{ has oscillatory behaviour} \end{cases}$
- if $K=0$ then both must be linear
- if $K<0$ then $\begin{cases} A \text{ has oscillatory behaviour} \\ B \text{ has exponential behaviour} \end{cases}$

- looking at the boundary conditions in x $(A(0)=A(L)=0)$ we see that if A is a linear combination of $e^{\sqrt{K}x}$ and $e^{-\sqrt{K}x}$ then the only solution is $A=0$

$$\rightarrow \quad K \leq 0$$

- We can rule out $K=0$ on the same ground

$\rightarrow$ $K<0$ so define $K=-k^2$

$\Rightarrow$ for each k,

$$A_k(x) = a_k \cos kx + b_k \sin kx$$

$$B_k(y) = \alpha_k e^{ky} + \beta_k e^{-ky}$$

or equivalently

$$= \tilde{\alpha}_k \cosh(ky) + \tilde{\beta}_k \sinh(ky)$$

$A_k(0) = A_k(L) = 0 \quad \Rightarrow \quad a_k = 0 \qquad k = \frac{n\pi}{L}$

$B_k(0) = 0 \quad \Rightarrow \quad \tilde{\alpha}_k = 0$

So $T(x,y) = \sum_{n=0}^{\infty}{}' \; b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$

To satisfy the bcs at $y=1$ we require that

$$T(x,1) = T_0(x) = \sum_{n=0}^{\infty}{}' \; b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi}{L}\right)$$

So $C_n = b_n \sinh\left(\frac{n\pi}{L}\right)$ are the Fourier coefficients of the periodic function constructed from $T_0(x)$ that is

- periodic with period $2L$
- antisymmetric at origin

<u>Homework</u>: Find the solution when

$$T_0(x) = \sin^2\left(\frac{\pi x}{L}\right)$$

<u>Note</u>

- We can see that if $T_0(x) = 0$ then the solution to the problem is the null function $T(x,y) = 0 \quad \forall x, y$.

$\rightarrow$ This is a property of elliptic equations: if the bcs are identically 0 on the contour of the domain then the solution to the Laplace equation (i.e. without source terms) is the null solution.

CHAPTER 5 : The 1D Wave equation

5.1 Canonical form and d'Alembert's solution

<u>Recall</u>: Given the 1D wave equation in the unbounded domain $\mathbb{R}$:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ \left.\dfrac{\partial u}{\partial t}\right|_{\substack{x \\ t=0}} = g(x) \end{cases}$$

- the change of variable

$$\xi = x + ct$$
$$\eta = x - ct$$

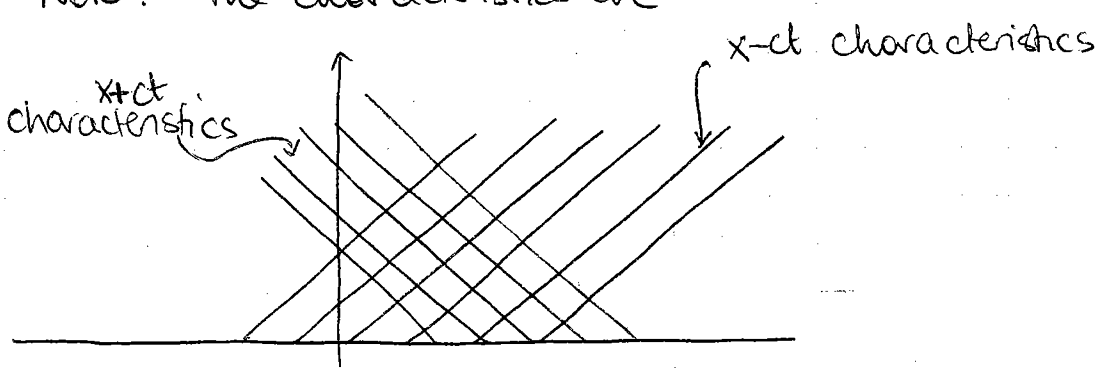
casts it in canonical form $u_{\eta\xi} = 0$

so that the solution can be written as the superposition of two functions

$$u = F(\xi) + G(\eta)$$

$$u(x,t) = F(x+ct) + G(x-ct)$$

- Note: the characteristics are



The solution at any point is the <u>sum</u> of the information carried by both characteristics intersecting at that point → no shock

• To determine the functions F and G we must satisfy the initial conditions:

$$\begin{cases} u(x,0) = F(x) + G(x) = f(x) & (1) \\ \frac{\partial u}{\partial t}(x,0) = +cF'(x) - cG'(x) = g(x) & (2) \end{cases}$$

Integrating (2) $\Rightarrow$

$$F(x) - G(x) = \int_0^x \left[F'(u) - G'(u)\right] du + F(0) - G(0)$$

$$= \int_0^x \frac{g(u)}{c}\, du + F(0) - G(0)$$

Solving with (1) $\Rightarrow$

$$2F(x) = \int_0^x \frac{1}{c} g(u)\, du + F(0) - G(0) + f(x)$$

$$2G(x) = f(x) - \int_0^x \frac{1}{c} g(u)\, du - F(0) + G(0)$$

So:
$$F(x) = \frac{1}{2} f(x) + \frac{1}{2}\left[F(0) - G(0)\right] + \frac{1}{2c}\int_0^x g(u)\, du$$
$$G(x) = \frac{1}{2} f(x) - \frac{1}{2}\left[F(0) - G(0)\right] - \frac{1}{2c}\int_0^x g(u)\, du$$

So
$$u(x,t) = F(x+ct) + G(x-ct)$$
$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du$$

$\rightarrow$ d'Alembert's formula for the homogeneous wave equation with constant velocity c.

<u>Note</u>: This solution works even if the initial conditions are discontinuous $\Rightarrow$ the wave equation <u>preserves</u> the discontinuities.

5.2 Examples

① Consider an infinitely long tube filled with air maintained at a constant temperature such that the sound speed in the tube is also constant

Pressure waves in the tube satisfy a 1D wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}$$

At $t=0$ a perturbation is created (by tapping on the tube symmetrically) such that

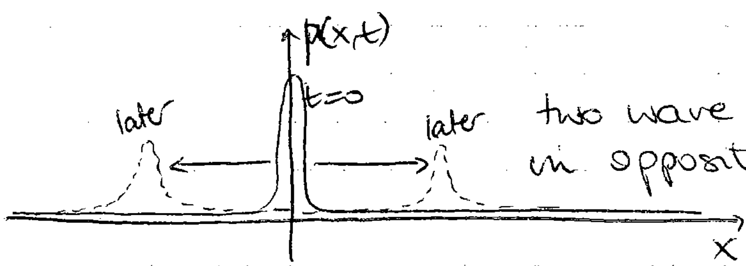
$$p(x,0) = p_0 e^{-x^2/2\sigma^2}$$

$$\frac{\partial p}{\partial t} = 0$$

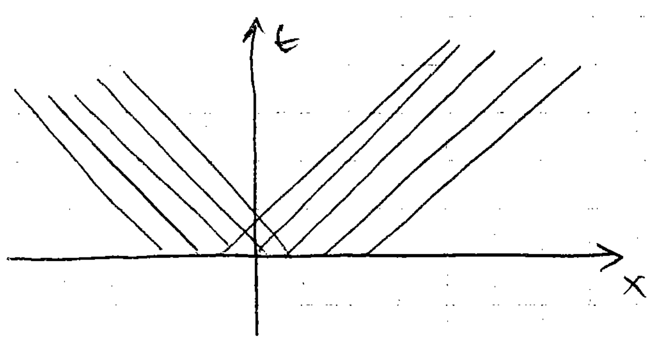
→ What is $p(x,t)$?

Solution Apply d'Alembert's formula

$$p(x,t) = \frac{p_0}{2}\left[e^{-\frac{(x-ct)^2}{2\sigma^2}} + e^{-\frac{(x+ct)^2}{2\sigma^2}}\right]$$



two wave packets are emitted travelling in opposite directions with velocity c.



Each set of characteristics carry half of the information away from the initial perturbation.

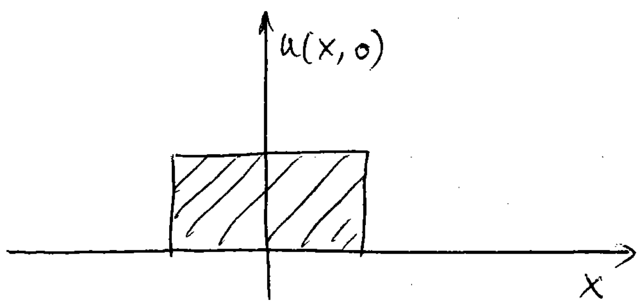
② What is the solution for $u_{tt} = c^2 u_{xx}$ and

$$\begin{cases} u(x,0) = 2 & \text{if } |x| \le a \\ u(x,0) = 0 & \text{if } |x| > a \\ \frac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

Again: use

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

with $\begin{cases} f(x) = 2 & \text{if } |x| \le a \\ f(x) = 0 & \text{if } |x| > a \end{cases}$



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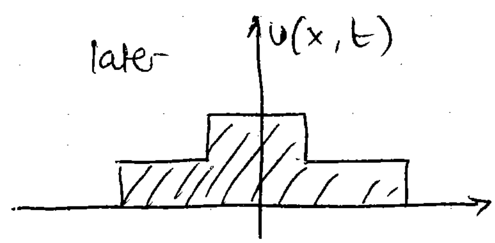

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