

Part A: Multiple choice questions

Remember to record your final answer in the table on page 2.

**Question 1:**[2 points] Find whether the following integral converges or diverges. If it converges, give the value of the integral.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx.$$

(A) 3

(B) 2/3

(C) 1/3

(D) -1/2

(E) It diverges

**Question 2:**[2 points] Find the value of the following definite integral:

$$\int_1^e x^2 \ln(x^3) dx.$$

(A) 1

(B)  $2/3e^3 - 1/3$

(C)  $2/3e^3 + 1/3$

(D)  $1/3e^3 + 1/3$

(E)  $1/3e^3 - 1/3$

**Question 3:**[2 points] Determine the volume of the solid obtained by rotating the region bounded by  $y = \sqrt{x}$ ,  $x = 0$ ,  $x = 4$  and the  $x$ -axis about the  $x$ -axis.

(A)  $16\pi$

(B)  $16\pi^2$

(C) 32

(D)  $32\pi^2$

(E)  $32\pi$

**Question 4:**[2 points] Find the solution of the following differential equation

$$\frac{dx}{dt} = 3\frac{x}{t^2}$$

with the initial condition  $x(1) = e^{-3}$ .

(A)  $x(t) = e^{-t}$

(B)  $x(t) = e^{-t/3}$

(C)  $x(t) = e^{-3/t}$

(D)  $x(t) = 1/t$

(E)  $x(t) = 2t^2 - 1$

**Question 5:**[2 points] Consider the following differential equation:

$$\frac{dx}{dt} = 2x^2 + x - 3^x.$$

Which one of the following statement is correct? (Only one statement is correct.)

(A) 1 is a stable equilibrium.

(B) 1 is an unstable equilibrium.

(C) 2 is a stable equilibrium.

(D) 2 is an unstable equilibrium.

① To find  $\int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx$  we calculate

$$\lim_{T \rightarrow 0^+} \int_T^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx.$$

But  $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx = 3\sqrt[3]{\sin x}$ . To see this pick  $u = \sin x$ .

Therefore  $\int_T^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx = 3 - 3\sqrt[3]{\sin T} \Rightarrow \int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx = 3.$

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② we have

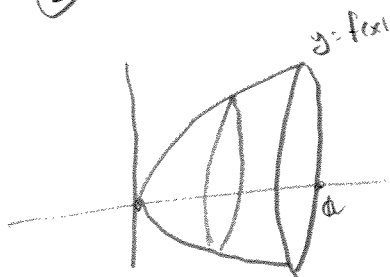
$$\int x^2 \ln(x^3) dx = x^3 \ln x - \frac{x^3}{3}.$$

To see this pick  $x^3 = u$  and notice that  $\int \ln x dx = x \ln x - x$ .

Therefore  $\int_1^e x^2 \ln(x^3) dx = (e^3 - \frac{e^3}{3}) - (-\frac{1}{3}) = \frac{2e^3}{3} + \frac{1}{3}.$

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③ we recall that



$$Vol = \pi \int_0^a f(x)^2 dx$$

so we have  $y = 2\sqrt{x}$  Therefore

$$Vol = \pi \int_0^4 4x dx = \frac{4\pi x^2}{2} \Big|_0^4 = 32\pi$$


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④ We have  $\frac{dx}{x} = \frac{30t}{t^2} \Rightarrow \ln x = \int \frac{30t}{t^2} \Rightarrow \ln(x) = -3/t + C,$   
 $\Rightarrow x(t) = ce^{-3/t}.$

But  $x(1) = e^{-3} \Rightarrow c = 1$ , therefore  $x(t) = e^{-3/t}$

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⑤ We have  $\frac{dx}{dt} = f(x)$  where  $f(x) = 2x^2 + x - 3^x$ .  $f(1) = 0$  so 1 is an equilibrium.  
 But  $f'(1) = 5 - 3 \log 3 > 0$  ( $f'(x) = 4x + 1 - \log_3 3^x$ ) so 1 is unstable.  
 (See Theorem 7.1 in the note Equilibrium).

(E) Neither 1 nor 2 is equilibrium.

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**Question 6:**[2 points] Consider the following matrices:

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 3 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 7 & 7 & 1 \end{pmatrix}$$

and the following expression:

- (I)  $AB$  is defined.
- (II)  $AC$  is defined. ✓
- (III)  $CA^T$  is defined. ✓
- (IV) The matrix  $B$  is invertible.
- (V)  $C + AA^T$  is defined.

Which one of the following statement is correct? (Only one statement is correct.)

- (A) (I), (III) and (V) are true.
  - (B) (I) and (IV) are true.
  - (C) Only (II) is true.
  - (D) (II) and (III) are true.
  - (E) (III) and (V) are true.
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**Question 7:**[2 points] Consider the following matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which number is not an eigenvalue of  $A$ .

- (A)  $1 + i$
  - (B)  $1 - i$
  - (C)  $1$
  - (D)  $i$
  - (E)  $1 + i$  and  $1 - i$ .
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**Question 8:**[2 points] Find the domain of

$$f(x, y) = \ln(x - y)$$

- (A)  $\{(x, y) \in \mathbb{R}^2 : x \geq y\}$ .
- (B)  $\{(x, y) \in \mathbb{R}^2 : x \leq y\}$ .
- (C)  $\{(x, y) \in \mathbb{R}^2 : x > y\}$ .



(D)  $\{(x, y) \in \mathbb{R}^2 : x < y\}$ .

(E)  $\{(x, y) \in \mathbb{R}^2 : x \neq y\}$ .

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**Question 9:**[2 points] Determine the tangent plane of

$$g(x, y) = \sin(xy),$$

at the point  $P = (1, 0)$ .

(A)  $z = x$ .

(B)  $z = y$ .

(C)  $z = y - 1$ .

(D)  $z = x - 1$ .

(E)  $z = x + y$ .

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**Question 10:**[2 points] Let

$$f(x, y) = (e^x \cos y, e^x \sin y).$$

Find the Jacobian matrix, evaluate it at  $(0, \pi)$ .

(A)  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(B)  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(C)  $\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$

(D)  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(E)  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



Part B: Long answer questions:

Question 11:[3 points] Calculate the following integral:

$$\int \frac{3x+1}{x^2-x-6} dx.$$

We have  $x^2-x-6 = (x-3)(x+2)$ . Therefore we need to find A and B so that

$$\frac{3x+1}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{(A+B)x + (2A-3B)}{x^2-x-6}$$

Therefore we have  $\begin{cases} A+B=3 \\ 2A-3B=1 \end{cases}$ . Hence

$$\left( \begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -3 & 1 \end{array} \right) \xrightarrow{-2R_1+R_2=R_2} \left( \begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -5 & -5 \end{array} \right) \Rightarrow \begin{cases} B=1 \\ A=2 \end{cases}$$

Hence

$$\int \frac{3x+1}{x^2-x-6} dx = \int \frac{2}{x-3} dx + \int \frac{1}{x+2} dx = 2\ln|x-3| + \ln|x+2| + C$$

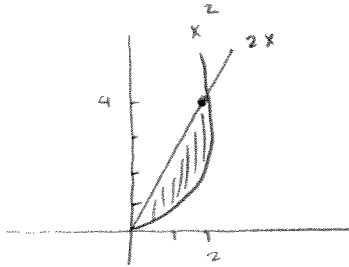
Final solution

$$2\ln|x-3| + \ln|x+2| + C$$



Question 12:[2 points] Find the area between  $f(x) = 2x$  and  $g(x) = x^2$  for  $0 \leq x \leq 2$ .

Since  $0 \leq x \leq 2$ , then  $x^2 \leq 2x$ . So we have



Therefore the area is

$$\int_0^2 (2x - x^2) dx = \int_0^2 2x - \int_0^2 x^2 dx$$
$$= x^2 \Big|_0^2 - \frac{1}{3} x^3 \Big|_0^2$$

$$= 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$

Final solution

$$\boxed{\frac{4}{3}}$$

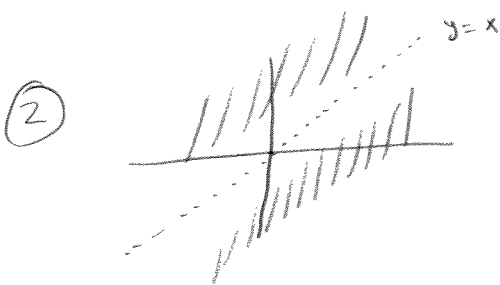


Question 13: Consider the following function of two variables

$$f(x, y) = \frac{1}{x-y}$$

1. (1 point) Find the domain of  $f(x, y)$ .
2. (2 points) Represent the domain of  $f(x, y)$  graphically in the plane.
3. (2 points) Find the level sets  $L_c$  of  $f$ . Plot these curves for  $c = 1$  and  $c = 1/2$ .

①  $\text{Dom}_f = \{ (x, y) \in \mathbb{R}^2 : x - y \neq 0 \} = \{ (x, y) \in \mathbb{R}^2 : x \neq y \}$

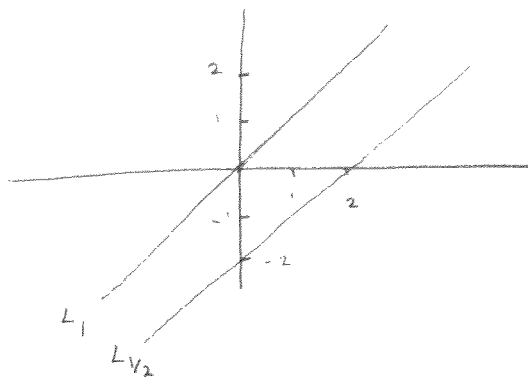


③  $L_c = \{ (x, y) \in \mathbb{R}^2 : f(x, y) = c \} = \{ (x, y) \in \mathbb{R}^2 : \frac{1}{x-y} = c \}$ .

so  $L_c$  is defined for all  $c \in \mathbb{R}$ . Now

①  $L_1 = \{ (x, y) \in \mathbb{R}^2 : \frac{1}{x-y} = 1 \} = \{ (x, y) \in \mathbb{R}^2 : x = y \}$

②  $L_{1/2} = \{ (x, y) \in \mathbb{R}^2 : \frac{1}{x-y} = \frac{1}{2} \} = \{ (x, y) \in \mathbb{R}^2 : y = x - 2 \}$





Question 14:[4 points] Find the explicit solution of the following linear system of differential equations

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

with initial conditions  $x(0) = 2$  and  $y(0) = 5$ .

Put  $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$ . We need to find the eigenvalues and eigenvectors of  $A$ . We have

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = (1-\lambda-2)(1-\lambda+2) \\ = (-\lambda-1)(-\lambda+3)$$

Therefore the eigenvalues of  $A$  are  $\lambda_1 = -1$ ,  $\lambda_2 = 3$ .

$\lambda_1 = -1$ : To find an eigenvector we need to solve  $AV = -V$

So we get  $AV + V = 0 \Rightarrow (A + I)V = 0$ . Hence we need to solve

$$\left( \begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left( \begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array} \right) \xrightarrow{-2R_1 + R_2 = R_2} \left( \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

So  $x + 2y = 0$ . Put  $y = 1$ , then  $x = -2$ . Therefore

$V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$  is an eigenvector.

$\lambda_2 = 3$ : To find an eigenvector we need to solve  $AV = 3V$ .

Therefore we need to solve  $(A - 3I)V = 0$ , so

$$\left( \begin{array}{cc|c} -2 & 4 & 0 \\ 1 & -2 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left( \begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array} \right) \xrightarrow{2R_1 + R_2 = R_2} \left( \begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

Therefore  $x - 2y = 0$ . Pick  $y = 1$  then  $x = 2$ . Therefore  $V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

is an eigenvector.

Hence the general solution of the equation is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now we find  $c_1$  and  $c_2$  we have  $x(0) = 2$  and  $y(0) = 5$ . Therefore

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\text{So } \begin{cases} -2c_1 + 2c_2 = 2 \\ c_1 + c_2 = 5 \end{cases} \Rightarrow \left( \begin{array}{cc|c} 1 & 1 & 5 \\ -2 & 2 & 2 \end{array} \right) \xrightarrow{2R_1 + R_2 = R_2} \left( \begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 4 & 12 \end{array} \right)$$

$\Rightarrow c_2 = 3, c_1 = 2$ . Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Question 15:[6 points] Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$

First we compute  $\det(A - \lambda I) = ?$  we have

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 2 \\ 1 & 3-\lambda & 1 \\ 2 & 0 & -\lambda \end{pmatrix} = (-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} + 2 \begin{vmatrix} 1 & 3-\lambda \\ 2 & 0 \end{vmatrix}$$

$$\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = -2$$

$$\begin{aligned} &= (-\lambda)(-\lambda(3-\lambda)) + 2(-2(3-\lambda)) \\ &= \lambda^2(3-\lambda) - 4(3-\lambda) \\ &= (3-\lambda)(\lambda^2 - 4) = (3-\lambda)(\lambda-2)(\lambda+2) \end{aligned}$$

$\lambda_1 = 3$ : we need to solve  $(A - 3I)v = 0$ . hence

$$\begin{pmatrix} -3 & 0 & 2 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 2 & 0 & -3 & | & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ -3 & 0 & 2 & | & 0 \\ 2 & 0 & -3 & | & 0 \end{pmatrix} \xrightarrow{\substack{3R_1 + R_2 = R_2 \\ -2R_1 + R_3 = R_3}} \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & 5 & | & 0 \\ 0 & 0 & -5 & | & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 + R_3 = R_3} \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & 5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}. \text{ Therefore } \begin{cases} z = 0, y \text{ is a free variable} \\ x = 0 \end{cases}$$

So pick  $y = 1$  then  $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  is an eigenvector.

$\lambda_2 = 2$ : we need to solve  $(A - 2I)v = 0$  hence

$$\begin{pmatrix} -2 & 0 & 2 & | & 0 \\ 1 & 1 & 1 & | & 0 \\ 2 & 0 & -2 & | & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ -2 & 0 & 2 & | & 0 \\ 2 & 0 & -2 & | & 0 \end{pmatrix} \xrightarrow{R_2 + R_3 = R_3} \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ -2 & 0 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{2R_1 + R_2 = R_2} \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 2 & 4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Hence  $x + y + z = 0$ ,  $y + 2z = 0$  and  $z$  is a free variable. Pick

$z = 1$  then  $y = -2$  and  $x = 1$ , so  $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$  is an eigenvector.

$\lambda_3 = -2$ : we need to solve  $(A + 2I)v = 0$  so

$$\left( \begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 1 & 5 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left( \begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array} \right) \xrightarrow{-R_2 + R_3 = R_3} \left( \begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{-2R_1 + R_2 = R_2} \left( \begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

so  $y = 0$ ,  $z$  is a free variable and  $x + z = 0$ . Pick  $z = -1$

then  $x = -1$ . Hence  $v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  is an

eigenvector.

Final solution

$$\lambda_1 = 3, v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2, v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -2, v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

**Question 16:** For the following function

$$f(x, y) = x^2 + y^2,$$

Find

1. (1 point)  $\frac{\partial f}{\partial x}(x, y)$ .
2. (1 point)  $\frac{\partial f}{\partial y}(x, y)$ .
3. (2 point) The Jacobi matrix of  $F(x, y) = (f(x, y), x)$ .
4. (2 point) The tangent plane of  $f(x, y)$  at the point  $P = (0, 0)$ .

①  $\frac{\partial f}{\partial x} = 2x$

②  $\frac{\partial f}{\partial y} = 2y$

③ we have

$$J_F(x, y) = \begin{pmatrix} 2x & 2y \\ 1 & 0 \end{pmatrix}$$

④ we recall the formula.  $z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$

But  $z_0 = f(0, 0) = 0$ ,  $\frac{\partial f}{\partial x}(0, 0) = 0$ ,  $\frac{\partial f}{\partial y}(0, 0) = 0$ .

Hence

$$z = 0$$

is the tangent plane of  $f(x, y) = x^2 + y^2$  at  $P = (0, 0)$ .



**Question 17:** A cup of tea, originally at  $95^\circ\text{C}$ , cools down according to Newton's Law of Cooling:

$$\frac{dH}{dt} = \alpha(H - A),$$

where  $H(t)$  is the tea temperature at time  $t$ ,  $\alpha$  is a negative constant, and  $A = 20^\circ\text{C}$  is the temperature of the surrounding air. Suppose that after 10 minutes, the tea temperature is  $65^\circ\text{C}$ .

1. (2 points) Find  $H(t)$ .
2. (1 points) Sketch the graph of temperature against time.
3. (1 points) Find  $\lim_{t \rightarrow \infty} H(t)$ .

① we need to solve  $\frac{dH}{dt} = \alpha(H - A)$ . we have

$$\frac{dH}{H - A} = \alpha dt \Rightarrow \int \frac{dH}{H - A} = \int \alpha dt \Rightarrow \ln|H - A| = \alpha t + C_1.$$

therefore  $H - A = ce^{\alpha t}$  where  $c = e^{C_1}$ . Hence

$$H(t) = ce^{\alpha t} + A$$

We know  $H(0) = 95$ ,  $A = 20$ ,  $H(10) = 65$ . We use

these to find  $c$  and  $\alpha$ .

we have  $95 = H(0) = c + A \Rightarrow c = 95 - A = 95 - 20 = 75$

so  $c = 75$ . Moreover we have

$$65 = H(10) = 75e^{10\alpha} + 20 \Rightarrow 45 = 75e^{10\alpha}$$

$$\Rightarrow \frac{3}{5} = e^{10\alpha}$$

$$\Rightarrow 10\alpha = \ln\left(\frac{3}{5}\right)$$

$$\Rightarrow \alpha = \frac{1}{10} \ln\left(\frac{3}{5}\right)$$

$$c = 75$$

$$\alpha = \frac{1}{10} \ln\left(\frac{3}{5}\right)$$

and then  $H(t) = 75e^{\alpha t} + 20$  where  $\alpha = \frac{1}{10} \ln\left(\frac{3}{5}\right)$ .

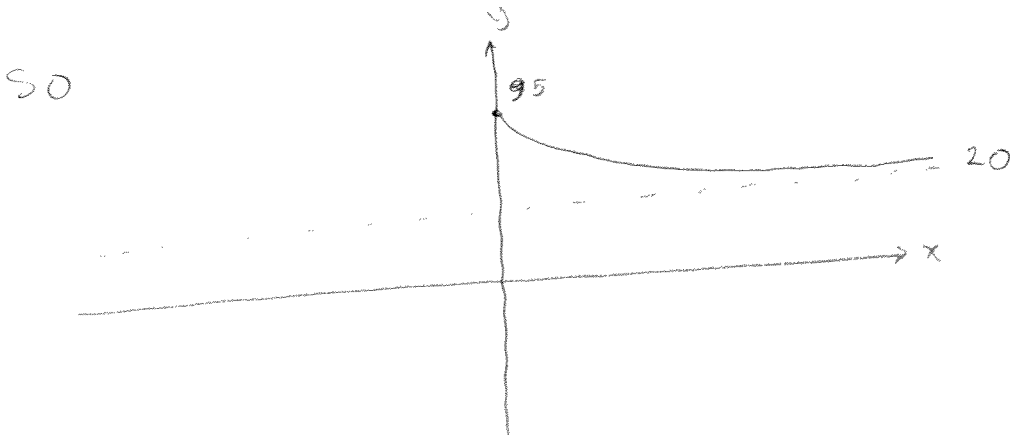
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Notice that  $\alpha < 0$ , since  $\ln\left(\frac{3}{5}\right) < 0$ .



(2) to plot the function  $H(t) = 75e^{\alpha t} + 20$  where  $\alpha = \frac{1}{10} \ln(\frac{3}{6}) < 0$

notice that  $\lim_{t \rightarrow \infty} H(t) = 20$  since  $\alpha < 0$  and  $\lim_{t \rightarrow \infty} e^{\alpha t} = 0$ .



(3)  $\lim_{t \rightarrow \infty} H(t) = 20$  since  $\lim_{t \rightarrow \infty} e^{\alpha t} = 0$  ( $\alpha < 0$ ).