This equality can only hold for all t, all x provided both sides are constant.

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = K \doteq \frac{c^2}{X(x)}\frac{d^2X}{dx^2}.$$

This constant is arbitrary for the moment, but it is an eigenvalue of the spatial operator $c^2\frac{d^2}{dx^2}$

$$c^2\frac{d^2X}{dx^2} = K\, X(x)$$

$\Rightarrow$ Will be determined by the boundary conditions.

We want to solve
$$\begin{cases} \dfrac{d^2X}{dx^2} = \dfrac{K}{c^2}X(x) \\ X(0) = X(L) = 0 \end{cases}$$

if $K>0$ then the solutions are exponential
$K<0$ ———————— oscillatory.

if $K>0$ then it is not possible to fit both bcs unless $X(x) = 0 \ \forall x$

so let's choose $K<0$, and write $\frac{K}{c^2} = -k^2$

$$\Rightarrow \quad X(x) = A\cos kx + B\sin kx$$

$$X(0) = 0 \Rightarrow A = 0$$
$$X(L) = 0 \Rightarrow \sin kL = 0 \Rightarrow k = \frac{n\pi}{L}$$

So there exist an ∞ of solutions for $X(x)$, namely

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad n \in \mathbb{Z}$$

Now we must solve $\frac{d^2T}{dt^2} = K\,T(t) = -c^2k^2\,T(t)$

so $\quad T(t) = a\cos(kct) + b\sin(kct)$

since there are an ∞ of possible values of k then for each $X_n(x)$

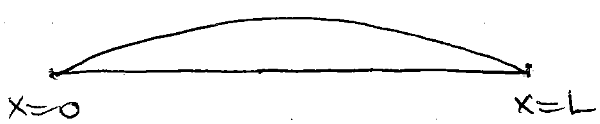
corresponds a $T_n(t) = a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)$

$\Rightarrow$ The general solution to the wave equation with these boundary conditions is a linear combination of all the solutions:

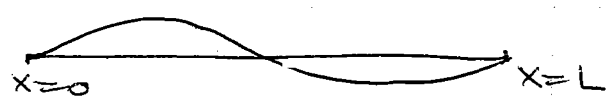
$$y(x,t) = \sum_n X_n(x)T_n(t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left[\alpha_n \cos\left(\frac{n\pi c t}{L}\right) + \beta_n \sin\left(\frac{n\pi c t}{L}\right)\right]$$

(incorporate a_n with A_n and b_n with B_n + any linear combination coefficient).

Note : • This expression shows that each spatial function $\sin\left(\frac{n\pi x}{L}\right)$ vibrates with its own frequency $\frac{nc}{2L}$ intrinsic to the system.



$n=1$
(0 nodes)



$n=2$
(1 node)

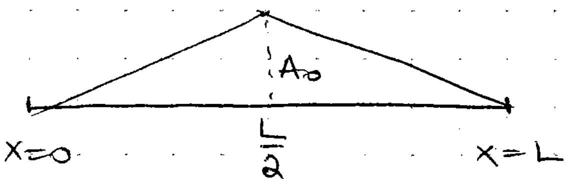
(4 nodes) $n=5$

Each mode has a different frequency. The higher the degree n of the mode, the higher the frequency (the pitch of the sound emitted for example).

• To determine which mode is excited and with which amplitude it is vibrating, we need to apply initial conditions to the system.

<u>Example</u>: Suppose we pluck the string in the middle, so that at $t=0$ we release it from rest with

$$y(x,0) = \begin{cases} \dfrac{2A_0}{L}x & \text{if } 0<x<\frac{L}{2} \\ 2A_0 - \dfrac{2x}{L}A_0 & \text{if } x\in\left[\frac{L}{2}, L\right] \end{cases}$$



$$\frac{\partial y}{\partial t}(x,0) = 0 \quad \text{since we release it from rest}$$

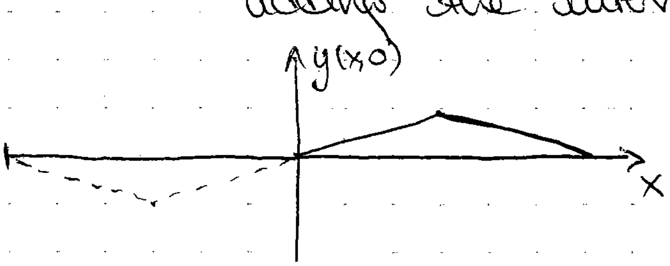
$\Rightarrow$ then we fit the general form $y(x,t)$ to these bcs by requiring that

$$y(x,0) = \sum_n{}' \alpha_n \sin\frac{n\pi x}{L} = \begin{cases} \dfrac{2A_0 x}{L} & \text{if } x\in\left[0,\frac{L}{2}\right] \\ 2A_0 - \dfrac{2A_0 x}{L} & \text{if } x\in\left[\frac{L}{2}, L\right] \end{cases}$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_n{}' \frac{n\pi c}{L}\beta_n \sin\left(\frac{n\pi x}{L}\right) = 0 \quad \Rightarrow \boxed{\beta_n = 0}$$

The first of these two equations implies that we are seeking the coefficients α_n such that the series on the left is equal to the function on the RHS $\Rightarrow$ this <u>looks</u> like a Fourier series problem!

<u>Problem</u>: $y(x,0)$ is not periodic $\Rightarrow$ to remedy the problem, turn $y(x,0)$ into a periodic function by adding the interval $[-L,0]$



$\rightarrow$ construct an odd function since we are looking for a sin expansion

$$\Rightarrow \text{By definition} \quad \alpha_n = \frac{1}{L}\int_{-L}^{L} y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\text{by symmetry} \quad = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\alpha_n = \frac{2}{L}\left\{ \int_0^{\frac{L}{2}} \left(\frac{2A_0 x}{L}\right)\sin\frac{n\pi x}{L}\,dx + \int_{\frac{L}{2}}^{L}\left(2A_0 - \frac{2A_0 x}{L}\right)\sin\frac{n\pi x}{L}\,dx \right\}$$

$$= \frac{2}{L}\left\{ \left[\frac{2A_0 x}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\right]_0^{\frac{L}{2}} + \int_0^{\frac{L}{2}} \frac{2A_0}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx \right.$$

$$+ 2A_0\left(-\frac{L}{n\pi}\right)\left[\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L} - \left[\frac{2A_0 x}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L}$$

$$\left. - \int_{\frac{L}{2}}^{L} \frac{2A_0}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx \right\}$$

$$= \frac{2}{L}\left\{ \left[-\frac{2A_0 x}{n\pi}\cos\frac{n\pi x}{L} + \frac{2A_0}{n\pi}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{\frac{L}{2}} \right.$$

$$\left. + \left[-\frac{2A_0 L}{n\pi}\cos\frac{n\pi x}{L} + \frac{2A_0 x}{n\pi}\cos\frac{n\pi x}{L} - \frac{2A_0}{n\pi}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L} \right\}$$

$$= \frac{2}{L}\left\{ -\cancel{\frac{2A_0}{n\pi}\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)} + \frac{2A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \right.$$

$$- \cancel{\frac{2A_0 L}{n\pi}\cos(n\pi)} + \cancel{\frac{2A_0 L}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \cancel{\frac{2A_0 L}{n\pi}\cos(n\pi)}$$

$$\left. - \cancel{\frac{2A_0}{n\pi}\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)} - \cancel{\frac{2A_0 L}{n^2\pi^2}\sin(n\pi)} + \frac{2A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{2}{L}\left\{ \frac{4A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \right\} = \frac{8A_0}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \qquad |n|>1$$

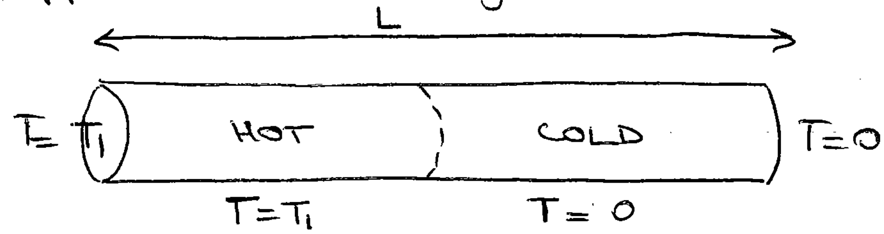
$\Rightarrow$ Finally,

$$y(x,t) = \sum_{n=1}^{\infty} \underbrace{\frac{8A_0}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)}_{\text{amplitude of the mode}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{spatial mode}} \underbrace{\cos\left(\frac{n\pi c t}{L}\right)}_{\text{temporal variation of the mode.}}$$

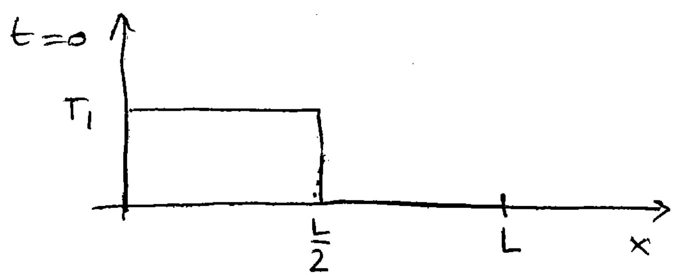
The solution is a superposition of all the modes vibrating independently with constant amplitude determined by their initial conditions.

4.2 Heat diffusion in a rod

Suppose we initially have a rod half-heated



The side walls are insulated so that heat can only be transferred laterally (x-direction).



The edges are kept at temperatures 0 and T_1 respectively

$$\begin{cases} T(0, t) = T_1 \\ T(L, t) = 0 \end{cases}$$

The PDE is $\dfrac{\partial T}{\partial t} = D \dfrac{\partial^2 T}{\partial x^2}$

Again, we try separating the variables such that

$$T(x, t) = A(x)\,B(t)$$

$$\Rightarrow \quad A(x)\frac{dB}{dt} = D B(t) \frac{d^2 A}{dx^2}$$

$$\Rightarrow \quad \frac{1}{B}\frac{dB}{dt} = \frac{D}{A}\frac{d^2 A}{dx^2} = \text{constant } K$$

So $$\begin{cases} \dfrac{dB}{dt} = KB \\[2mm] \dfrac{d^2 A}{dx^2} = \dfrac{KA}{D} \end{cases}$$

→ as before we expect K to be negative to satisfy the boundary conditions simultaneously, so that $K = -k^2$.

$\rightarrow$ for each value of k there is a possible solution

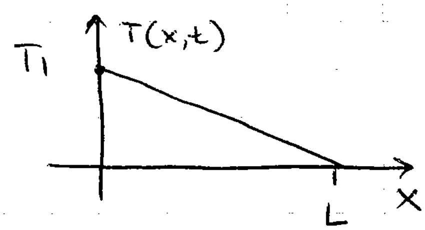
$$A_k(x) = \alpha_k \cos\left(\frac{k}{\sqrt{D}}x\right) + \beta_k \sin\left(\frac{k}{\sqrt{D}}x\right), \quad B_k(t) = e^{-k^2 t}$$

Important Note: if $K = 0$ then there is also a solution with

$$A = ax + b, \quad B = \text{constant}$$

To fit the boundary conditions, let us use our intuition about the problem

- we expect that as $t \to \infty$ the system relaxes to a temperature profile



$$T(x, t\to\infty) = T_1 - \frac{T_1}{L}x$$

$\rightarrow$ that's the $ax+b$ part!

- The behaviour of $\frac{dB}{dt} = -k^2 B$ suggests decaying exponential modes for all $k \neq 0$

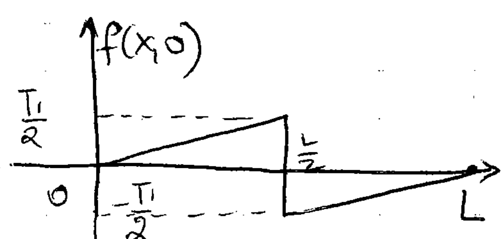
$\Rightarrow$ We expect the solution to be

$$T(x,t) = T_1 - \frac{T_1}{L}x + \begin{pmatrix}\text{some spatial} \\ \sin/\cos \text{ mode}\end{pmatrix} \cdot \begin{pmatrix}\text{a decaying} \\ \text{exponential}\end{pmatrix}$$

$$= T_1 - \frac{T_1}{L}x + f(x,t)$$

where $\begin{cases} f(0,t) = 0 \\ f(L,t) = 0 \end{cases}$

and $f(x,0) = T(x,0) - \left[T_1 - \frac{T_1}{L}x\right]$.



$\rightarrow$ Now we see that if $f(0,t) = 0$ then $\alpha_k = 0$ and if $f(L,t) = 0$ then

$$\frac{k}{\sqrt{D}}L = n\pi \Rightarrow k = \frac{n\pi\sqrt{D}}{L}$$