COMPLEX NUMBERS

A complex number z is of the form

$z = a + ib$, where $i^2 = -1$, and $a, b \in R$.

a=real part of z, a =Re z.
b=imaginary part of z, b =Im z.
z is real $\iff b = 0$.
z is purely imaginary $\iff a = 0$.

Let $z = a + ib$ and $w = c + id$. Then,

$$z + w = a + c + i(b + d)$$

$$z - w = a - c + i(b - d)$$

$$\begin{aligned} z \cdot w &= (a + ib) \cdot (c + id) \\ &= ac + iad + ibc + i^2bd \\ &= ac - bd + i(ad + bc) \end{aligned}$$

$$kz = ka + i(kb), \ k \in R.$$

If $z = a + ib$ is any complex number, then the complex conjugate of z is denoted by $\overline{z}$ (read "z bar") is defined by $\overline{z} = a - ib$.

$$\overline{z+w} = \overline{z} + \overline{w} \qquad \overline{z-w} = \overline{z} - \overline{w}$$

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \qquad \overline{\overline{z}} = z$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}} \qquad z + \overline{z} = 2 \text{ Re } z$$

$$z - \overline{z} = 2i \text{ Im } z$$

Let $z = a + ib$ and $w = c + id \neq 0$.

$$\begin{aligned}
\frac{z}{w} &= \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} \\
&= \frac{ac+bd+i(bc-ad)}{c^2+d^2} \\
&= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} = x + iy
\end{aligned}$$

where $x = \dfrac{ac+bd}{c^2+d^2}$ and $y = \dfrac{bc-ad}{c^2+d^2}$.

The absolute value (modulus) of $z = a + ib$ is

$$|z| = \sqrt{z\,\bar{z}} = \sqrt{a^2 + b^2}$$

We have the following equalities:

$$|zw| = |z| \cdot |w|, \qquad z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

Example: Let $z = 9 - 8i$ and $w = 5 + 2i$. Then find $|z|$, $|w|$, $|z/w|$.

Write $\frac{z}{w}$ in the form of $a + ib$.

Solution:

$$|z| = \sqrt{9^2 + (-8)^2} = \sqrt{81 + 64} = \sqrt{145}$$

$$|w| = \sqrt{5^2 + 2^2} = \sqrt{25 + 4} = \sqrt{29}$$

$$|z/w| = \frac{|z|}{|w|} = \frac{\sqrt{145}}{\sqrt{29}} = \frac{\sqrt{5 \cdot 29}}{\sqrt{29}} = \sqrt{5}$$

$$\frac{z}{w} = \frac{9 - 8i}{5 + 2i} = \frac{9 - 8i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i}$$

$$= \frac{(45 - 16) + i(-40 - 18)}{25 + 4} = 1 - 2i$$

Homework

1. Let $z = 3 + 4i$ and $w = 5 - 2i$. Express the followings in the form of $a + ib$.

(i) $(z - w)^2$ **(ii)** $\frac{z}{w}$ **(iii)** $\frac{\overline{z}}{\overline{w}}$ **(iv)** $\frac{1}{z^2}$ **(v)** $\frac{w}{2z}$

2. Find:

(i) Re $\frac{1}{2+i}$ **(ii)** Im $\frac{2+i}{3+4i}$ **(iii)** Im $\frac{2-i}{3-4i}$

3. Write the followings in the form of $a + ib$.

(i) $\frac{11+2i}{4+3i}$ **(ii)** $(3 + 5i)(3 - 5i)$

(iii) $(7 - 3i) - (-2 + 4i)$ **(iv)** $\frac{6+i}{7+3i}$

(v) $\frac{1}{(3+4i)^2}$ **(vi)** $\frac{\sqrt{3}+i}{(1-i)(\sqrt{3}-i)}$

4. In each part solve for z.

(i) $iz = 2 - i$ **(ii)** $(4 - 3i)\overline{z} = i$

5. If $z = 1 - 5i$ and $w = 3 + 4i$, find

$|z|$, $|w|$, $|z/w|$, $|\overline{z/w}|$, and $|\overline{z}/\overline{w}|$,

Trigonometric Ratios of Important Angles

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\pi/2$	1	0	$-$

$$(1 \text{ radian} = \frac{180^0}{\pi} = 57^0).$$

Polar Form of a Complex Number

Let $z = a + ib$.

$$\begin{aligned}
\cos\theta &= \frac{a}{|z|} \Longrightarrow a = |z|\cos\theta \\
\sin\theta &= \frac{b}{|z|} \Longrightarrow b = |z|\sin\theta \\
z &= a + ib = |z|\cos\theta + i|z|\sin\theta \\
&= |z|(\cos\theta + i\sin\theta) = |z|\text{cis}\theta.
\end{aligned}$$

Here θ is the angle between the positive real axis and the point z, $-\pi < \theta \leq \pi$
(all angles are measured in radians).
θ is called the argument of z, and it is denoted by $\theta = \arg z$.

$$\boxed{z = |z|(\cos\theta + i\sin\theta)}$$

is called the polar form of z.

Example: Find the polar form of $z = 1 + i$.

Solution: $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{|z|} = \frac{1}{\sqrt{2}} \\ \sin\theta &= \frac{b}{|z|} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \Longrightarrow \theta = \pi/4.$$

$$z = |z|(\cos\theta + i\sin\theta) = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$$

Example: What is the polar form of $z = 3 + i3\sqrt{3}$?

Solution: $|z| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{36} = 6.$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{|z|} = \frac{3}{6} = \frac{1}{2} \\ \sin\theta &= \frac{b}{|z|} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \end{aligned}\right\} \Longrightarrow \theta = \pi/3.$$

$$z = |z|(\cos\theta + i\sin\theta)$$

$$z = 6(\cos\pi/3 + i\sin\pi/3) = 6\mathrm{c}i\mathrm{s}(\pi/3)$$

Example: What is the polar form of $z = \sqrt{2} - i\sqrt{2}$?

Solution: $|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = 2.$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{|z|} = \frac{\sqrt{2}}{2} \\ \sin\theta &= \frac{b}{|z|} = \frac{-\sqrt{2}}{2} \end{aligned}\right\} \Longrightarrow \theta = -\pi/4.$$

$z = |z|(\cos\theta + i\sin\theta) = 2(\cos\pi/4 - i\sin\pi/4)$

Homework

1) Write the polar form of the following complex numbers:

(i) $z = -4 + 4i$ **(ii)** $z = 4i$

✓**(iii)** $z = -7$ **(iv)** $z = \dfrac{2+2i}{1-i}$, **(v)** $z = 1$.

2) Represent in the form of $a + ib$:

(i) $z = 4(\cos\pi/2 + i\sin\pi/2)$

(ii) $z = \sqrt{8}(\cos\pi/4 + i\sin\pi/4)$

✓**(iii)** $2\text{cis}(-\pi/6)$ **(iv)** $\dfrac{2\ \text{cis}\ (-3\pi/4)}{2\ \text{cis}\ (5\pi/6)}$.

Complex Division in Polar Form

If $z_1 = r_1$ cis θ_1 and $z_2 = r_2$ cis θ_2, then

$$\frac{z_1}{z_2} = \frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2),$$

and

$$\overline{z_1} = r_1 \text{ cis } (-\theta_1)$$

(complex conjugate of z in polar form).

Example: $z =$ cis $(\pi/2)$ and $w = 2$ cis $(-\pi/3)$. Find z/w, $\overline{z}$ and $\overline{w}$.

Solution:

$$\begin{aligned} \frac{z}{w} &= \frac{\text{cis } (\pi/2)}{2 \text{ cis } (-\pi/3)} \\ &= \frac{1}{2} \text{ cis } (\pi/2 - (-\pi/3)) = \frac{1}{2} \text{ cis } (5\pi/6) \\ &= \frac{1}{2}(\cos(5\pi/6) + i\sin(5\pi/6)) \\ &= \frac{1}{2}(\frac{-\sqrt{3}}{2} + i\frac{1}{2}) = \frac{-\sqrt{3}}{4} + i\frac{1}{4} \end{aligned}$$

$$\overline{z} = \text{cis}(-\pi/2), \quad \overline{w} = 2\text{cis}(\pi/3).$$

Complex Multiplication in Polar Form

If $z_1 = |z_1| \text{ cis } \theta_1$ and $z_2 = |z_2| \text{ cis } \theta_2$, then

$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot \text{ cis } (\theta_1 + \theta_2)$.

Example: If $z = 2 \text{ cis } \frac{3\pi}{8}$ and $w = 5 \text{ cis } \frac{2\pi}{3}$.

Then

$$\begin{aligned} z \cdot w &= \left(2 \text{ cis } \frac{3\pi}{8}\right)\left(5 \text{ cis } \frac{2\pi}{3}\right) \\ &= 2 \cdot 5 \text{ cis } \left(\frac{3\pi}{8} + \frac{2\pi}{3}\right) \\ &= 10 \text{ cis } \left(\frac{25\pi}{24}\right) \end{aligned}$$

De Moivre's Theorem:

For any positive integer n,

$$z^n = (|z|(\cos\theta + i\sin\theta))^n = |z|^n(\cos n\theta + i\sin n\theta).$$

Example: Write $z = (1+i)^{20}$ in the form of $a + ib$.

Solution: $1 + i = \sqrt{2}$ cis $(\pi/4)$. Hence

$$\begin{aligned}
(1+i)^{20} &= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{20} \\
&= (\sqrt{2})^{20}\left(\cos\frac{20\pi}{4} + i\sin\frac{20\pi}{4}\right) \\
&= 2^{10}(\cos(5\pi) + i\sin(5\pi)) \\
&= 2^{10}(\cos(4\pi+\pi) + i\sin(4\pi+\pi)) \\
&= 2^{10}(\cos\pi + i\sin\pi) \\
&= 2^{10}(-1 + i\cdot 0) \\
&= -2^{10} = -1024
\end{aligned}$$

Homework: Express the following complex numbers in the form of $a + ib$.

1) $z = (2 \text{ cis } (\pi/3))^6$. Ans: 64.

2) $z = (-1 + i)^4$.

3) $z = (1 - i)^{10}$. Ans: $-32i$

4) $z = (1 - i)^{27}$. Ans: $-2^{13}(1 + i)$

5) $z = (1 + i)^{12}$. Ans: -64

6) $z = (1 - i)^6(\sqrt{3} + i)^3$.

7) $z = (\sqrt{3} - i)^9(2 - 2i)^5$.

Roots of a Complex Number

Let $z^n = \alpha \text{ cis}\theta$. Then

$$\boxed{z_k = \sqrt[n]{\alpha} \text{ cis}\left(\frac{\theta + 2k\pi}{n}\right)};$$

where $k = 0, 1, 2, ..., n-1$.

Example: Let $z^3 = -8i$. Find z and write it in the standard form.

Solution: $\alpha = |-8i| = 8,\ \theta = -\pi/2$.

$$z_k = \sqrt[3]{8} \text{ cis}\left(\frac{-\pi/2 + 2k\pi}{3}\right);\ k = 0, 1, 2.$$

$$\begin{aligned}
z_0 &= 2 \text{ cis}\left(\frac{-\pi/2}{3}\right) = 2 \text{ cis } \left(\frac{-\pi}{6}\right) \\
&= 2 \ (\cos(-\pi/6) + i\sin(-\pi/6)) \\
&= 2 \ \left(\frac{\sqrt{3}}{2} + i\frac{-1}{2}\right) = \sqrt{3} - i \\
z_1 &= 2 \text{ cis}\left(\frac{(-\pi/2) + 2\pi}{3}\right) = 2 \text{ cis } (\pi/2) \\
&= 2 \ (\cos(\pi/2) + i\sin(\pi/2)) = 2(0 + i) = 2i.
\end{aligned}$$

$$
\begin{aligned}
z_2 &= 2 \text{ cis}\left(\frac{(-\pi/2)+4\pi}{3}\right) \\
&= 2 \text{ cis } (7\pi/6) \\
&= 2 \ (\cos(7\pi/6) + i \sin(7\pi/6)) \\
&= 2 \ (\frac{-\sqrt{3}}{2} - i\frac{1}{2}) = -\sqrt{3} - i.
\end{aligned}
$$

Example: Find the roots of $z^2 + z + 1 = 0$.
Solution:

$$z^2 + z + 1 = (z + \frac{1}{2})^2 - (\frac{1}{2})^2 + 1 = 0$$

$$\Longrightarrow \quad (z + \frac{1}{2})^2 + \frac{3}{4} = 0$$

$$\Longrightarrow \quad (z + \frac{1}{2})^2 = \frac{-3}{4}$$

$$\Longrightarrow \quad z + \frac{1}{2} = \pm\frac{\sqrt{3}}{2}i$$

$$\Longrightarrow \quad z = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\text{i.e, } z_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i \ , \quad z_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$

Homework: Find all complex numbers such that:

1) $z^2 = i$.

2) $z^3 = i$.

3) $z^3 = -i$.

4) $z^3 = 27i$.

✓ 5) $z^3 = -27i$.

6) $z^3 = 64i$.

7) $z^4 = -1$.

8) $z^4 = 2(i\sqrt{3} - 1)$.

9) $z^6 = -64$.

Answers:

1) $z_0 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $z_1 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

2) $z_0 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, $z_1 = \frac{-\sqrt{3}}{2} + i\frac{1}{2}$, $z_2 = -i$.

3) $z_0 = i$ $z_1 = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$, $z_2 = \frac{\sqrt{3}}{2} - i\frac{1}{2}$.

4) $z_0 = -3i$, $z_1 = -\frac{3\sqrt{3}}{2} + i\frac{3}{2}$, $z_2 = \frac{3\sqrt{3}}{2} + i\frac{3}{2}$.

5) $z_0 = 3i$, $z_1 = -\frac{3\sqrt{3}}{2} - i\frac{3}{2}$, $z_2 = \frac{3\sqrt{3}}{2} - i\frac{3}{2}$.

6) $z_0 = -4i$, $z_1 = -2\sqrt{3} + 2i$, $z_2 = 2\sqrt{3} + 2i$.

8) $\sqrt{2}\ e^{\pi i/6}$, $\sqrt{2}\ e^{4\pi i/6}$, $\sqrt{2}\ e^{7\pi i/6}$, $\sqrt{2}\ e^{10\pi i/6}$.