2.5.4 Crossing of characteristics ; initial shock position (x_c, t_c)

For conservation laws $\begin{cases} u_t + [F(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases}$

the characteristics are straight lines with $t = \dfrac{x-s}{F'(\phi(s))}$

Two lines emanating from s_1 and s_2 on the initial condition curve have equations

$$\begin{cases} t = \dfrac{x-s_1}{F'(\phi(s_1))} & \Rightarrow \quad x = tF'(\phi(s_1)) + s_1 \\[2ex] t = \dfrac{x-s_2}{F'(\phi(s_2))} & \Rightarrow \quad x = tF'(\phi(s_2)) + s_2 \end{cases}$$

$\Rightarrow$ they intercept at time t_+ satisfying

$$t_+ F'(\phi(s_1)) + s_1 = t_+ F'(\phi(s_2)) + s_2$$

$$\Leftrightarrow \quad t_+ = \frac{s_2 - s_1}{F'(\phi(s_1)) - F'(\phi(s_2))}$$

So t_c is the minimum value of t_+ over all possible values of s_1 and s_2 (such that $t_+ \geq 0$)

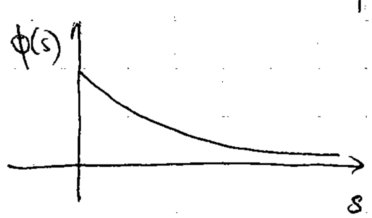
$$t_c = \min_{s_1, s_2} \frac{s_2 - s_1}{F'(\phi(s_1)) - F'(\phi(s_2))} \quad \text{with } t_c \geq 0 .$$

<u>Note</u> : IF $F'(\phi(s_1)) \leq F'(\phi(s_2))$ while $s_2 \geq s_1$ then t_c cannot be ≥ 0

$\Longrightarrow$ if $\dfrac{d}{ds}\left[F'(\phi(s))\right] > 0$ for all s then characteristics never intercept for $t \geq 0$.

<u>Example 1</u> Traffic flow: $F'(\phi(s)) = V_0\left(1 - \dfrac{2\phi(s)}{V_{max}}\right)$

- if $\phi(s) = \dfrac{V_{max}}{4} e^{-s}$



$$F'(\phi(s)) = V_0\left(1 - \frac{e^{-s}}{2}\right)$$

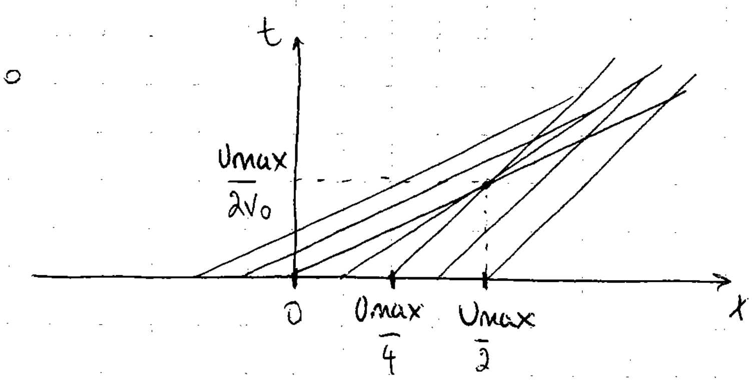
so $\dfrac{d}{ds}\left[F'(\phi(s))\right] = \dfrac{V_0}{2} e^{-s} \geq 0 \Rightarrow$ no shock.

Example 2

$$\phi(s) = \begin{cases} 0 & \text{if} \quad s \leq 0 \\ s & \text{if} \quad 0 \leq s \leq \frac{U_{max}}{4} \\ \frac{U_{max}}{4} & \text{if} \quad s \geq \frac{U_{max}}{4} \end{cases}$$

$$\Rightarrow \quad F'(\phi(s)) = \begin{cases} V_0 & \text{if} \quad s \leq 0 \\ V_0\left(1 - \frac{2s}{U_{max}}\right) & \text{if} \quad 0 \leq s \leq \frac{U_{max}}{4} \\ V_0/2 & \text{if} \quad s \geq \frac{U_{max}}{4} \end{cases}$$

So



Can we confirm this mathematically?

Select $s_2 > s_1$

- if $s_2 > s_1$, $s_1 \leq 0$ and $s_2 \leq 0 \Rightarrow$ no crossing

- if $s_2 > s_1$, $s_1 \leq 0$, $s_2 \in [0, \frac{U_{max}}{4}]$ then

$$t_+ = \frac{s_2 - s_1}{V_0 - V_0\left(1 - \frac{2s_2}{U_{max}}\right)} = \frac{s_2 - s_1}{\frac{2V_0 s_2}{U_{max}}} = \frac{U_{max}}{2V_0}\,\frac{s_2 - s_1}{s_2}$$

this is minimized when $s_1 \to 0$

- if $s_2 > s_1$ $\quad s_1 \in [0, \frac{U_{max}}{4}]$, $s_2 \geq \frac{U_{max}}{4}$ then

$$t_+ = \frac{s_2 - s_1}{V_0\left(1 - \frac{2s_1}{U_{max}}\right) - \frac{V_0}{2}} = \frac{s_2 - s_1}{\frac{V_0}{2} - \frac{2V_0 s_1}{U_{max}}}$$

this is minimized when $s_2 \to \frac{U_{max}}{4}$

$$\rightarrow \quad t_c = \min_{s_1 s_2} t_+ = \frac{U_{max}}{2V_0} \qquad x_c = F'(\phi(0))\,t_c = \frac{U_{max}}{2}$$ as seen on diagram.

2.5.5 Traffic flow revisited

From the example above, we now try to solve for $\gamma(t)$:

$$\frac{d\gamma}{dt} = \frac{F(u_+) - F(u_-)}{u_+ - u_-} \qquad \text{where } F(u) = V_0 u \left(1 - \frac{u}{U_{max}}\right)$$

- Characteristics from the left have $u_- = 0$

  ——————————— right ——— $u_+ = U_{max}/4$

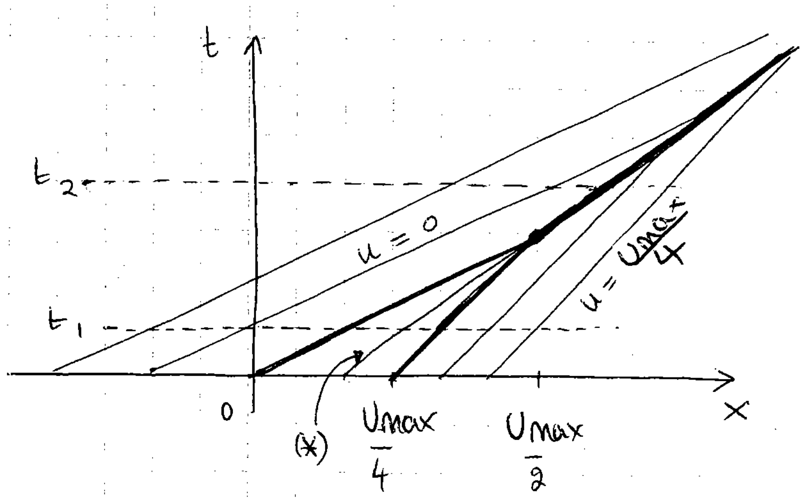
So $$\frac{d\gamma}{dt} = \frac{V_0 \frac{U_{max}}{4}\left(1 - \frac{1}{4}\right) - 0}{\frac{U_{max}}{4} - 0} = \frac{3V_0}{4}$$

So $$\gamma(t) - x_c = \frac{3V_0}{4}(t - t_c)$$

$$\Rightarrow \gamma(t) = \frac{U_{max}}{2} + \frac{3V_0}{4}\left(t - \frac{U_{max}}{2V_0}\right)$$

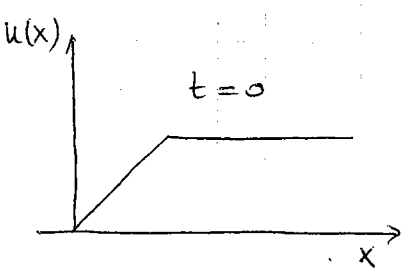
$$= \frac{3V_0}{4} t + \frac{U_{max}}{8}$$

So now we can construct the solution (graphically)

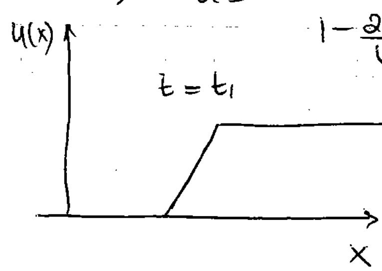


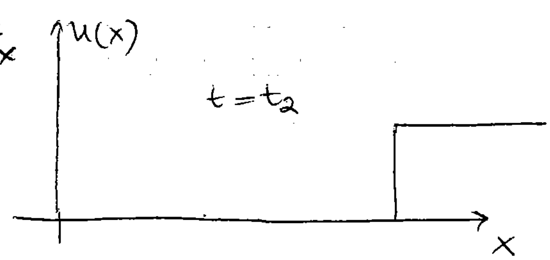
in (*): $$u = \phi(x - F'(\phi(\,))t) = x - V_0\left(1 - \frac{2u}{U_{max}}\right)t$$

So $$u\left(1 - \frac{2V_0}{U_{max}} t\right) = x - V_0 t$$



$$\Rightarrow u = \frac{x - V_0 t}{1 - \frac{2V_0 t}{U_{max}}}$$



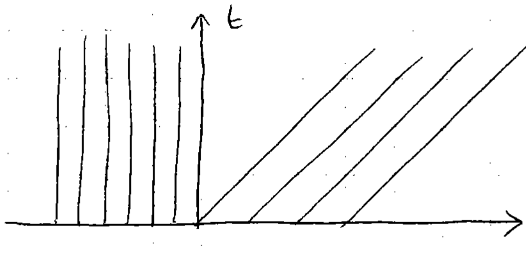


2.6.6 Expansion shocks and entropy condition

Consider the example

$$\begin{cases} u_t + u u_x = 0 & (F(u) = \tfrac{1}{2}u^2) \\ u(x,0) = \Theta(x) & \leftarrow \text{Heaviside function: } \begin{cases} \Theta(x) = 0 & x \leq 0 \\ \Theta(x) = 1 & x \geq 0 \end{cases} \end{cases}$$

Characteristics



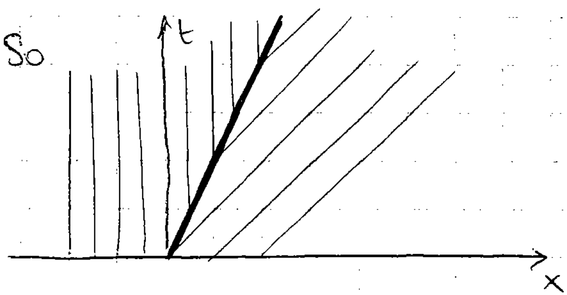
$$\begin{cases} t = \tau \\ x = u_0(s)\tau + s \\ u = u_0(s) = \Theta(s) \end{cases}$$

We could consider constructing a weak solution $u = u_- = 0$ on the left of $\gamma(t)$ and $u = u_+ = 1$ on the right of $\gamma(t)$.

The R.H. condition implies $\dfrac{d\gamma}{dt} = \dfrac{F(u_+) - F(u_-)}{u_+ - u_-} = \dfrac{1}{2}$

$\Rightarrow$ here $\gamma(t) = \frac{1}{2}t$

So



$$\begin{cases} u = 0 & x \leq t/2 \\ u = 1 & x > t/2 \end{cases}$$

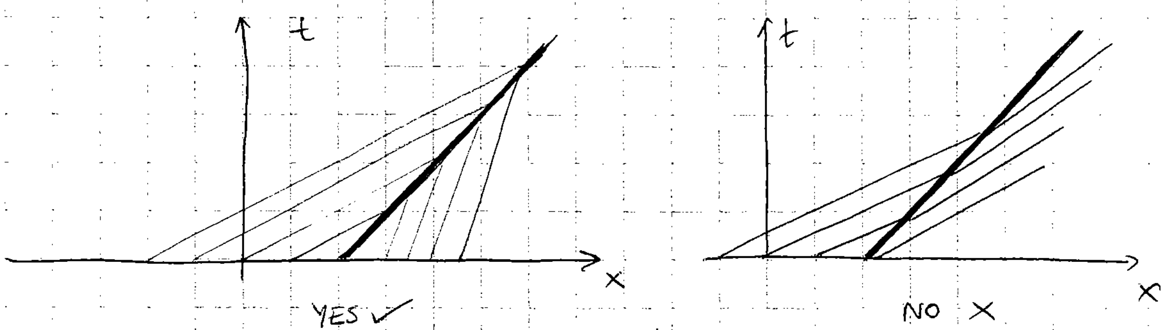
Problem: this solution is not physically acceptable because it is not causal: information appears to be "created" on the discontinuity and is then carried by the characteristics.

In other words we would like the system to be entirely determined by its initial conditions, not by arbitrary extensions of the solution

Definition The entropy condition

Characteristics must enter the discontinuity (the shock front) but are not allowed to emanate from it

To guarantee this, the slope of the characteristics on the left must be shallower than $\delta(t)$, and those on the right steeper.



$$\Rightarrow \quad \frac{1}{F'(u_-)} \leq \frac{1}{\frac{d\delta}{dt}} \leq \frac{1}{F'(u_+)}$$

$$\Leftrightarrow \quad \boxed{F'(u_-) \geq \frac{d\delta}{dt} \geq F'(u_+)}$$

Problem: How do we construct solutions if $F'(u)$ is an increasing function of u? (see example above).

Going back to the characteristic solutions:

$$\begin{cases} t = \tau \\ x = F'(\phi(s))t + s \\ u = \phi(s) \end{cases} \quad \text{to} \quad \begin{cases} u_t + [F(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Assume characteristics diverge from $s = s_0$. At this point,

$$x = F'(\phi(0))t = F'(u)t + s_0$$

so let's construct

$$u = G\left(\frac{x - s_0}{t}\right)$$

where G is the inverse function of F'.

and use this as a solution in the "fan" region

__Example 1__ $u_t + u u_x = 0$ $\quad F(u) = \frac{1}{2}u^2 \quad F'(u) = u$

with $u(x,0) = \Theta(x)$

Characteristics diverge from $s=0$, so that

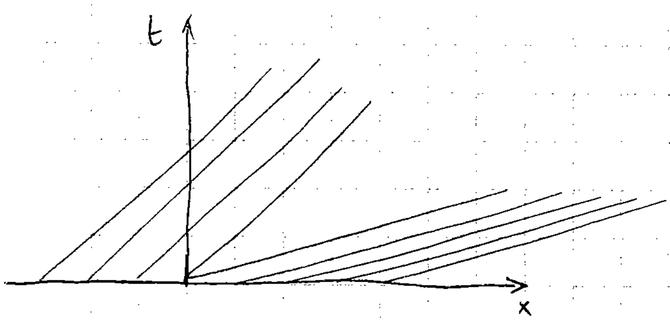
$$x = ut \quad \text{or} \quad u = x/t \quad \text{in the "fan"}$$

$\Rightarrow$ we construct the weak solution with

$$\begin{cases} u = 0 & \text{if } x \leq 0 \\ u = x/t & \text{if } 0 \leq x \leq t \\ u = 1 & \text{if } x \geq t \end{cases}$$

__Example 2__: $u_t + (e^u)_x = 0$ with $u(x,0) = \Theta(x)$

then $u_t + e^u u_x = 0$



$\rightarrow$ Characteristics are

$$t = \frac{x-s}{e^{\Theta(s)}} = \begin{cases} \frac{x-s}{1} & \text{if } s \leq 0 \\ \frac{x-s}{e} & \text{if } s \geq 0 \end{cases}$$

So let's construct from $x = e^u t + s$ the solution

$u = \ln(x/t)$ emanating from $s=0$

$$\Rightarrow \quad \begin{cases} u = 0 & \text{if } x \leq t \\ u = \ln(x/t) & \text{if } t \leq x \leq et \\ u = 1 & \text{if } x \geq et \end{cases}$$

__Check__: in region $t \leq x \leq et$

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2} \cdot \frac{t}{x} = -\frac{1}{t} \qquad e^u \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(e^u\right) = \frac{\partial}{\partial x}\left(\frac{x}{t}\right) = \frac{1}{t}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(e^u\right) = 0 \quad \text{as required}$$

2.6 Method of characteristics for fully nonlinear equations

(see handout for justification of method).

let $F(x, y, u, u_x, u_y) = 0$ be a first order PDE

let
$$p = u_x$$
$$q = u_y$$

then the new characteristic equations are

$$\frac{\partial x}{\partial \tau} = \frac{\partial F}{\partial p}$$

$$\frac{\partial y}{\partial \tau} = \frac{\partial F}{\partial q}$$

$$\frac{\partial u}{\partial \tau} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}$$

$$\frac{\partial p}{\partial \tau} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial u}$$

$$\frac{\partial q}{\partial \tau} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial u}$$

with initial conditions
$$x(s, \tau=0) = x_0(s)$$
$$y(s, \tau=0) = y_0(s)$$
$$u(s, \tau=0) = u_0(s)$$

and where $p_0(s)$ and $q_0(s)$ are solutions to the system of equations

$$\begin{cases} \dfrac{du_0}{ds} = p_0(s)\dfrac{dx_0}{ds} + q_0(s)\dfrac{dy_0}{ds} \\ F(x_0, y_0, u_0, p_0, q_0) = 0 \end{cases}$$

Note: • The condition for existence & uniqueness of solution is the same as for quasilinear equation.

• In the case where the system is quasilinear, the characteristic equations reduce to

$$\frac{\partial x}{\partial \tau} = a(x,y,u) \qquad \frac{\partial u}{\partial \tau} = c(x,y,u)$$

$$\frac{\partial y}{\partial \tau} = b(x,y,u)$$

as expected.

Example: The Eikonal equation

$$u_x^2 + u_y^2 = n^2(x, y)$$

- equation for the "propagation" of a wave in a medium of refraction index $n(\underset{\sim}{x})$ ($n = \frac{c_0}{c(x)}$ where c_0 = average wave velocity, $c(x)$ = local wave velocity), when the wavelength of the wave is $\ll$ typical lengthscale of variation of $n(\underset{\sim}{x})$.

  → can be derived from asymptotic analysis of the complete wave equation.

- Very commonly used in <u>geometrical optics</u>.

- The surfaces u = constant are wavefronts.

⇒ Here $F(x, y, u, p, q) = 0$

is $p^2 + q^2 - n^2(\underset{\sim}{x}) = 0$.

Take $n(\underset{\sim}{x}) = n_0$ for simplicity

The characteristic equations are

$$\frac{\partial x}{\partial \tau} = \frac{\partial F}{\partial p} = 2p$$

$$\frac{\partial y}{\partial \tau} = \frac{\partial F}{\partial q} = 2q$$

$$\frac{\partial u}{\partial \tau} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} = 2p^2 + 2q^2 = -2n_0^2$$

$$\frac{\partial p}{\partial \tau} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial u} = 0$$

$$\frac{\partial q}{\partial \tau} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial u} = 0$$

⇒ very easy to integrate

$$x = 2p\tau + x_0(s) \qquad u = -2n_0^2\tau + u_0(s)$$

$$y = 2q\tau + y_0(s) \qquad p = p_0(s)$$

$$q = q_0(s)$$

Case 1 $\quad u(x,2x) = 1$

$\Rightarrow$ initial condition curve is

$$x_0(s) = s$$
$$y_0(s) = 2s$$
$$u_0(s) = 1$$

while $p_0(s)$ and $q_0(s)$ satisfy

$$\begin{cases} p_0^2(s) + q_0^2(s) = n_0^2 \\ \dfrac{du_0}{ds} = p_0(s)\dfrac{dx_0}{ds} + q_0(s)\dfrac{dy_0}{ds} \end{cases}$$

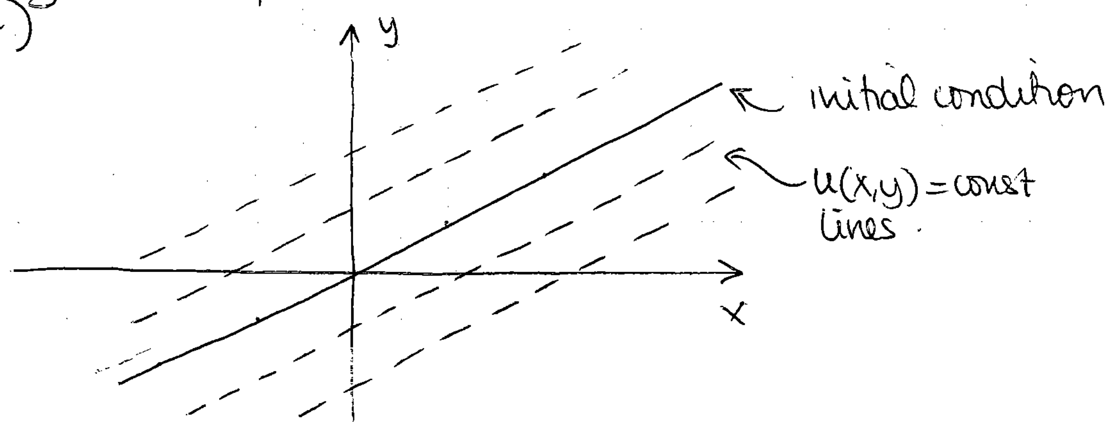
$$\Rightarrow \begin{cases} p_0^2(s) + q_0^2(s) = n_0^2 \\ p_0(s) + 2q_0(s) = 0 \end{cases} \Rightarrow \begin{cases} q_0^2(s) = \dfrac{n_0^2}{5} \\ p_0(s) = -2q_0(s) \end{cases}$$

Take $p_0(s) = 2\dfrac{n_0}{\sqrt{5}}$ and $q_0(s) = -\dfrac{n_0}{\sqrt{5}}$

So
$$\begin{cases} x = \dfrac{2n_0}{\sqrt{5}}\tau + s \\ y = -\dfrac{n_0}{\sqrt{5}}\tau + 2s \\ u = -2n_0^2\tau + 1 \end{cases} \Rightarrow \begin{cases} s = \dfrac{x+2y}{5} \\ \tau = \dfrac{\sqrt{5}}{2n_0}\left(\dfrac{4x-2y}{5}\right) = \dfrac{x-2y}{\sqrt{5}\,n_0} \end{cases}$$

so
$$u = -\frac{2n_0}{\sqrt{5}}(x-2y) + 1$$

Lines of constant u are <u>straight lines</u> parallel to $x = 2y$ (i.e. parallel to the initial condition curve)



<u>Case 2</u> $\quad u(\cos(s), \sin(s)) = 1 \qquad s \in [0, 2\pi]$

this time $\begin{cases} x_0(s) = \cos(s) \\ y_0(s) = \sin(s) \\ u_0(s) = 1 \end{cases}$

and $\begin{cases} p_0^2(s) + q_0^2(s) = n_0^2 \\ 0 = -\sin(s)\, p_0(s) + \cos(s)\, q_0(s) \end{cases}$

so $\begin{cases} p_0(s) = n_0 \cos(s) \\ q_0(s) = n_0 \sin(s) \end{cases}$

$$\begin{cases} x = 2n_0\cos(s)\tau + \cos(s) = (2n_0\tau+1)\cos(s) \\ y = 2n_0\sin(s)\tau + \sin(s) = (2n_0\tau+1)\sin(s) \\ u = -2n_0^2\tau + 1 \end{cases}$$

so $\quad x^2+y^2 = (2n_0\tau+1)^2 \Rightarrow \tau = \frac{1}{2n_0}\left(\sqrt{x^2+y^2} - 1\right)$

and $\quad u = -n_0\left(\sqrt{x^2+y^2} - 1\right) + 1$

this time lines of constant u (wave fronts) are circles