2.5 Weak solutions, shocks and entropy condition

2.5.1 Example of Burger's equation

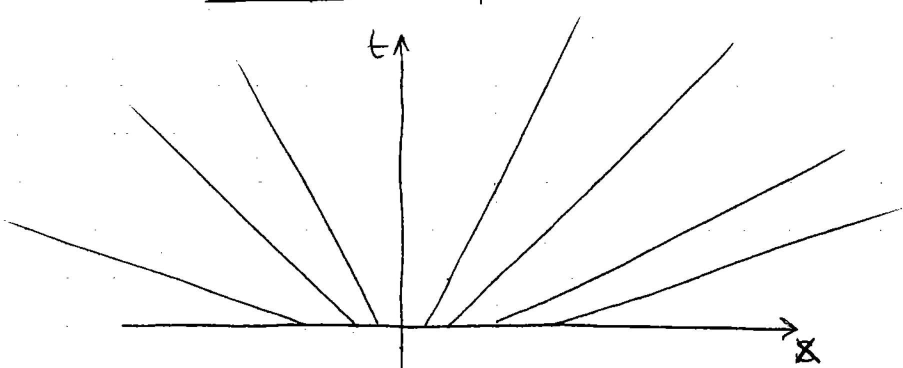
$$u_t + uu_x = 0$$
$$u(x,0) = f(x)$$

Characteristic equations:

$$\frac{dt}{d\tau} = 1 \quad \rightarrow \quad t = \tau$$
$$\frac{dx}{d\tau} = u \quad \rightarrow \quad x = u\tau + s$$
$$\frac{du}{d\tau} = 0 \quad \rightarrow \quad u = u_0(s) = f(s)$$

Characteristics: Straight lines $\quad t = \dfrac{x-s}{f(s)}$

Example 1: $f(s) = s$



Characteristics

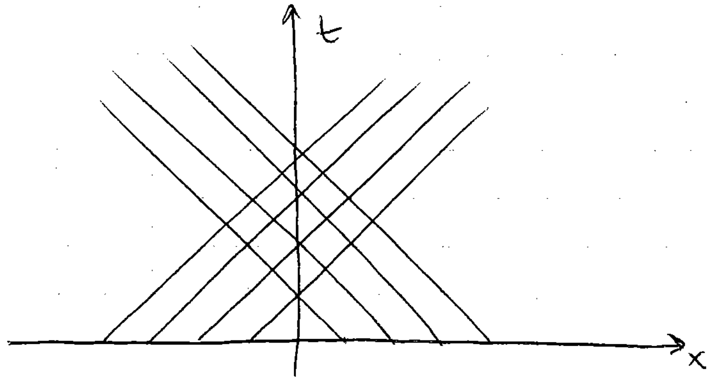
→ the solution exists at all times, no problem

$$u = f(s) = s$$
$$= x - ut$$

so $\quad u = \dfrac{x}{1+t}$

Example 2: First type of problem crossing characteristics

$$f(s) = -\frac{|s|}{s} = \begin{cases} 1 & \text{if} \quad x < 0 \\ -1 & \text{if} \quad x \geq 0 \end{cases}$$

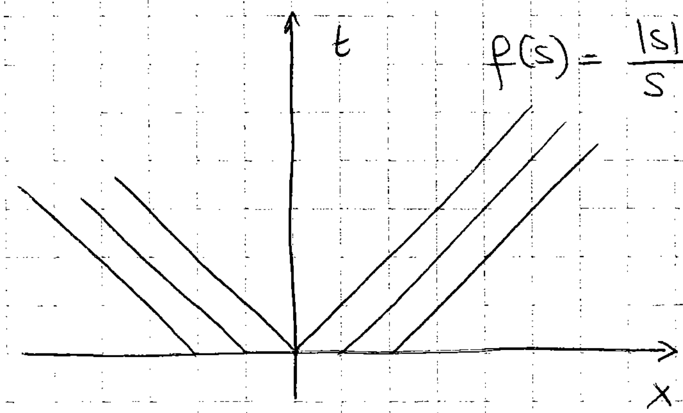


characteristics intercect!

Since $u = u_0(s)$ is constant on characteristics, which value should we choose?!

Example 3 Second type of problem: some region of space/time is not represented by any characteristic

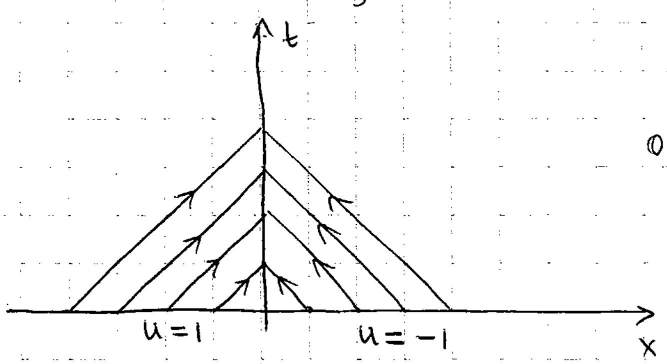
$$f(s) = \frac{|s|}{s} = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



What values should $u(x,t)$ take in the region which is not spanned by any characteristics?

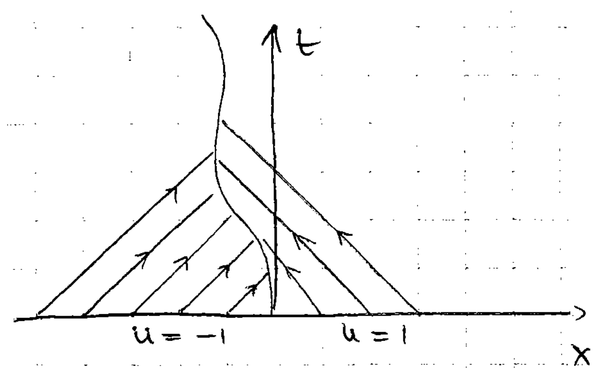
2.5.2 Weak problems and weak solutions

One way to resolve the first type of problem is to select a particular line separating characteristics emanating from the left & from the right and selecting the corresponding solution on each side



selecting $x=0$ line

or



selecting a random line ...

Problems!
- the solution then appears to be discontinuous at the line ⇒ SHOCK
- there is more than one solution

Non-smooth solutions are called weak solutions. Weak solutions are not solutions of the PDE since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ are not defined at discontinuities.
Weak solutions are solutions of the associated weak problem

<u>Definition</u>: A weak problem is an <u>integral</u> reformulation of the PDE for which solutions can be discontinuous!

<u>Note</u>: there are many possible weak problems associated to a given PDE.

2.5.3 Weak problems and conservation laws

Conservation laws of the kind

$$\frac{\partial u}{\partial t} + \nabla \cdot F = 0$$

are usually derived in physical systems from integral relationships anyway →

$$\frac{\partial}{\partial t}\int_{\text{volume}} u\,dV + \int_{\text{surface}} F \cdot dS = 0 = \frac{\partial}{\partial t}\int_{\text{volume}} u\,dV + \int_{\text{volume}} \nabla \cdot F\,dV$$

so we may as well use these integral formulations as our weak problem.

Take $\quad u_t + \frac{\partial}{\partial x}\left[F(u)\right] = 0$

and integrate over an interval $[a,b]$ at a given time t:

$$\frac{\partial}{\partial t}\int_a^b u\,dx + \int_a^b \frac{\partial}{\partial x}\left[F(u)\right] dx = 0$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t}\int_a^b u\,dx + F(u(b,t)) - F(u(a,t)) = 0 \qquad (*)$$

→ this is the weak formulation of a conservation law

- Any smooth solution of $(*)$ is also a solution of the associated PDE.

- However, we can now construct non-smooth solutions.

Assume the solution has one discontinuity in the solution $u(x,t)$ located on the line $x = \gamma(t)$ such that

$$\begin{cases} u(x,t) = u_-(x,t) & \text{if} \quad x < \gamma(t) \\ u(x,t) = u_+(x,t) & \text{if} \quad x > \gamma(t) \end{cases}$$

Then, plugging this into $(*)$ we get

$$\frac{\partial}{\partial t}\left[\int_a^{\gamma(t)} u_-(x,t)\,dx + \int_{\gamma(t)}^b u_+(x,t)\,dx\right] + F(u(b,t)) - F(u(a,t)) = 0$$

Recall : $$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \frac{db}{dt} f(b(t),t) - \frac{da}{dt} f(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx$$

So

$$\frac{\partial}{\partial t}\int_a^{\gamma(t)} u_-(x,t)\,dx = \frac{d\gamma}{dt}\, u_-(\gamma(t),t) + \int_a^{\gamma} \frac{\partial u_-}{\partial t}\,dx$$

$$\frac{\partial}{\partial t}\int_{\gamma(t)}^b u_+(x,t)\,dx = -\frac{d\gamma}{dt}\, u_+(\gamma(t),t) + \int_{\gamma(t)}^b \frac{\partial u_+}{\partial t}\,dx$$

So $$\frac{d\gamma}{dt}\left[u_-(\gamma(t),t) - u_+(\gamma(t),t)\right] + \int_a^{\gamma(t)} \frac{\partial u_-}{\partial t}\,dx + \int_{\gamma(t)}^b \frac{\partial u_+}{\partial t}\,dx$$
$$+ F(u(b,t)) - F(u(a,t)) = 0$$

Now write

$$\begin{aligned} F(u(b,t)) - F(u(a,t)) &= F(u(b,t)) - F(u_+(\gamma(t),t)) + F(u_+(\gamma(t),t)) \\ &\quad + F(u_-(\gamma(t),t)) - F(u(a,t)) - F(u_-(\gamma(t),t)) \\ &= \int_a^{\gamma(t)} \frac{\partial}{\partial x}(F(u_-))\,dx + \int_{\gamma(t)}^b \frac{\partial}{\partial x}(F(u_+))\,dx + F(u_+(\gamma(t),t)) \\ &\quad - F(u_-(\gamma(t),t)) \end{aligned}$$

So finally we get

$$\cancel{\int_a^{\gamma(t)} \frac{\partial u_-}{\partial t} + \frac{\partial}{\partial x}\left(F(u_-)\right) dx} + \cancel{\int_{\gamma(t)}^b \frac{\partial u_+}{\partial t} + \frac{\partial}{\partial x}\left(F(u_+)\right) dx}$$

$$+ \frac{d\gamma}{dt}\Big(u_-(\gamma(t),t) - u_+(\gamma(t),t)\Big) + F(u_+(\gamma(t),t)) - F(u_-(\gamma(t),t)) =$$

$$\Rightarrow \qquad \frac{d\gamma}{dt} = \frac{F(u_+(\gamma(t),t)) - F(u_-(\gamma(t),t))}{u_+(\gamma(t),t) - u_-(\gamma(t),t)}$$

An equation for the discontinuity curve (shock curve) in terms of the jump in u and $F(u)$ across the shock. Sometimes written as

$$\frac{d\gamma}{dt} = \frac{[F]}{[u]} \qquad \text{Rankine - Hugoniot jump condition}$$

To find $\gamma(t)$ we need an initial condition: take $\gamma(t_c) = x_c$ where t_c is the earliest time (with $t_c > 0$) for which characteristics cross and x_c is the position at which this happens.

Example 1 Burgers equation $\left(F(u) = \frac{u^2}{2}\right)$ with $f(s) = -\frac{|s|}{s}$.

We saw the earliest (positive) characteristics crossing occurs at $x_c = 0$, $t_c = 0$

On the left side of $\gamma(t)$, $u = u_- = 1$

— right —— $u = u_+ = -1$

$F(u_+) = \frac{1}{2}$ $\quad F(u_-) = \frac{1}{2}$ so

$$\frac{d\gamma}{dt} = \frac{0}{2} = 0 \Rightarrow \gamma = \text{constant} \Rightarrow \gamma = 0$$

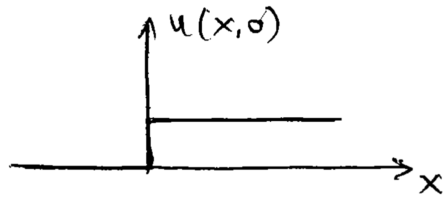
So the correct discontinuity line is $\boxed{x = 0}$.

Example 2 Traffic problems:

Suppose we try to solve for a traffic flow

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F(u)) = 0 \quad \text{with } F(u) = V_0 u\left(1 - \frac{u}{U_{max}}\right)$$

$$u(x,0) = \begin{cases} 0 & \text{if } x<0 \\ \frac{U_{max}}{4} & \text{if } x>0 \end{cases}$$

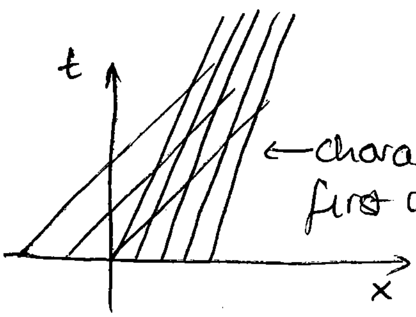


The characteristics are given by

$$x = F'(\phi(s))t + s \quad \text{where } \begin{array}{l} \phi(s) = 0 \text{ if } s<0 \\ \phi(s) = \frac{U_{max}}{4} \text{ if } s>0 \end{array}$$

so

$$\begin{cases} x = V_0 t + s & \text{if } s<0 \\ x = \frac{V_0}{2}t + s & \text{if } s>0 \end{cases}$$



← characteristics first cross at $x=0$, $t=0$

on left: $u_- = 0$

on right: $u_+ = \frac{U_{max}}{4}$

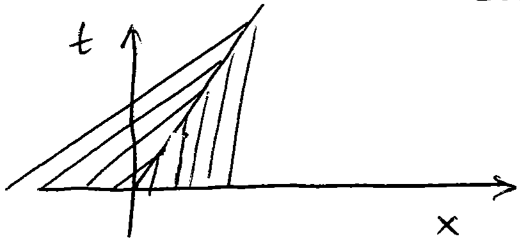
$F(u_-) = 0$

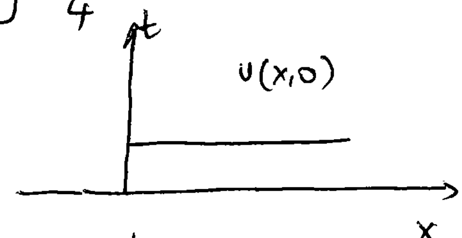
$F(u_+) = V_0 \cdot \frac{U_{max}}{4}\left(1 - \frac{1}{4}\right) = V_0 \frac{3U_{max}}{16}.$

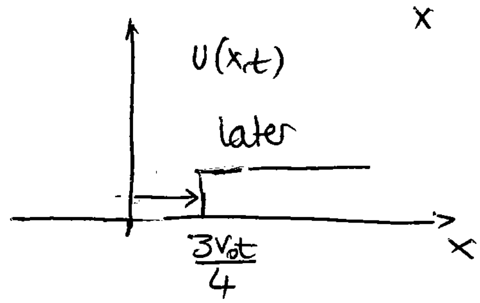
$$\text{so} \quad \frac{d\gamma}{dt} = \frac{[F]}{[u]} = \frac{3V_0}{4}$$

$$\Rightarrow \text{since } \gamma(t=0) = 0 \text{ then } \gamma(t) = \frac{3V_0 t}{4}$$

So the shock travels at velocity $\frac{3V_0}{4}$







Homework

Repeat same problem with

$$u(x,0) = \begin{cases} 0 & \text{if } x<0 \\ \frac{3U_{max}}{4} & \text{if } x>0 \end{cases}.$$