2.4 Examples of use of quasilinear 1st order PDEs

2.4.1 Conservation laws (general)

- Conservation laws are usually written in the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(F(u,x,t)\right) = 0$$

(or, in more than one dimension

$$\frac{\partial u}{\partial t} + \nabla\cdot\left(F(u,\underline{r},t)\right) = 0 \; . \,)$$

- Why is this called a conservation law?

$\Rightarrow$ Expressed in their integral form, the conservation laws describes the conservation of the quantity u:

$$\int_D \frac{\partial u}{\partial t}\, d^3\underline{r} + \int_D \nabla\cdot\left(F(u,\underline{r},t)\right) d^3\underline{r} = 0$$

$$\Leftrightarrow \quad \underbrace{\frac{\partial}{\partial t}\int_D u\, d^3\underline{r}}_{\text{total change of the quantity } u \text{ in the domain } D} + \underbrace{\int_{S = \text{contour of } D} F(u,\underline{r},t)\, d^2\underline{r}}_{\text{Flux through the surface of the domain}} = 0$$

Example: The standard equation for the conservation of mass in a flow stirred by a velocity field $v(\underline{r})$ is

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\, v(\underline{r})\right) = 0$$

It can be rewritten as

$$\int_D \frac{\partial \rho}{\partial t} d\underline{r}^3 + \int_D \nabla \cdot (\rho v(r))\, d^3\underline{r}$$

$$\Leftrightarrow \quad \frac{\partial m}{\partial t} + \int_{\substack{\text{surface} \\ \text{of } D}} \rho v(\underline{r})\, d^2\underline{r} = 0$$

change of mass within volume D

mass carried by velocity field across surface of D

- Conservation laws occur in most domains in science.

2.4.2 Conservation law (specific)

Here for simplicity we consider conservation laws which can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(F(u)\right) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

(the flux is a function of u only)

Then we see that

$$\frac{\partial u}{\partial t} + F'(u)\frac{\partial u}{\partial x} = 0$$

$\rightarrow$ a quasilinear homogeneous PDE that can be integrated with

$$\begin{cases} \dfrac{\partial t}{\partial \tau} = 1 & \rightarrow \quad t = \tau \\ \dfrac{\partial x}{\partial \tau} = F'(u) & (*) \\ \dfrac{\partial u}{\partial \tau} = 0 & \rightarrow \quad u \text{ is constant along characteristics:} \\ & \quad u = u_0(s) = \phi(s) \end{cases}$$

So from $(*)$ we see that

$$\frac{\partial x}{\partial \tau} = F'(\phi(s)) \implies x = F'(\phi(s))\,\tau + x_0(s)$$

in other words

$$\left\{\begin{array}{l} \tau = t \\ x = F'(\phi(s))t + s \\ u = \phi(s) \end{array}\right\} \text{ so } \quad u = \phi(x - F'(u)t)$$

Interpretation

① u is constant on characteristics

② The characteristics are straight lines in the $(x-t)$ plane with slope $\frac{1}{F'(\phi(s))}$, which depends only on the initial condition (and on F'.)

③ The problem of finding $u(x,t)$ becomes equivalent to solving the algebraic equation

$$u = \phi\left(x - F'(u)t\right)$$

It is sometimes possible to invert this analytically.

2.4.3 Burger's inviscid equation (Euler's equation)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$\Leftrightarrow \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \quad \rightarrow \text{ a conservation law with } F(u) = \frac{u^2}{2}$$

$$F'(u) = u$$

so the solution for any given initial condition $u(x,0) = \phi(x)$ can formally be written as

$$u(x,t) = \phi\left(x - u(x,t)\,t\right)$$

which may or may not be solvable analytically

① Suppose

$$\phi(x) = 3x \quad \text{then}$$

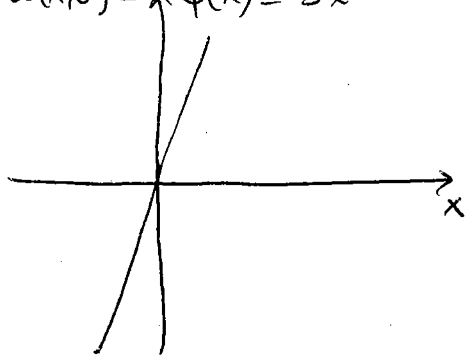
$$u(x,t) = 3\left(x - u(x,t)t\right)$$

$$\Leftrightarrow \quad u(x,t)\left[1 + 3t\right] = 3x$$

$$\Rightarrow \quad u(x,t) = \frac{3x}{1+3t}$$

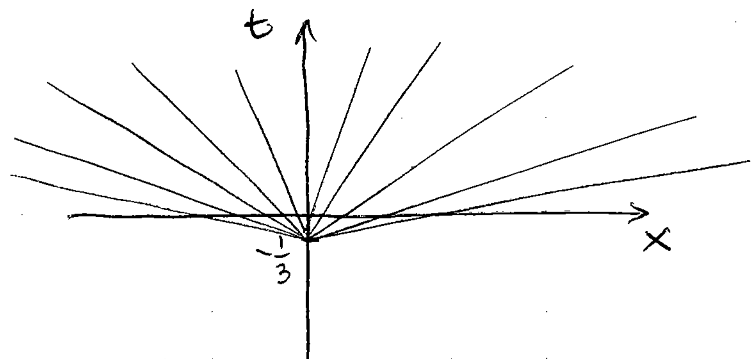
<u>Interpretation</u>

$u(x,0) = \phi(x) = 3x$



Characteristics are lines in the $(x$-$t)$ plane with slope $\frac{1}{F'(\phi(s))}$

Here the slope is $\frac{1}{3s}$



$$C^{(s)} : x = 3s \cdot t + s$$

$$\Leftrightarrow \quad t = \frac{x-s}{3s}$$

(they all pass through the point $(0, -\frac{1}{3})$)

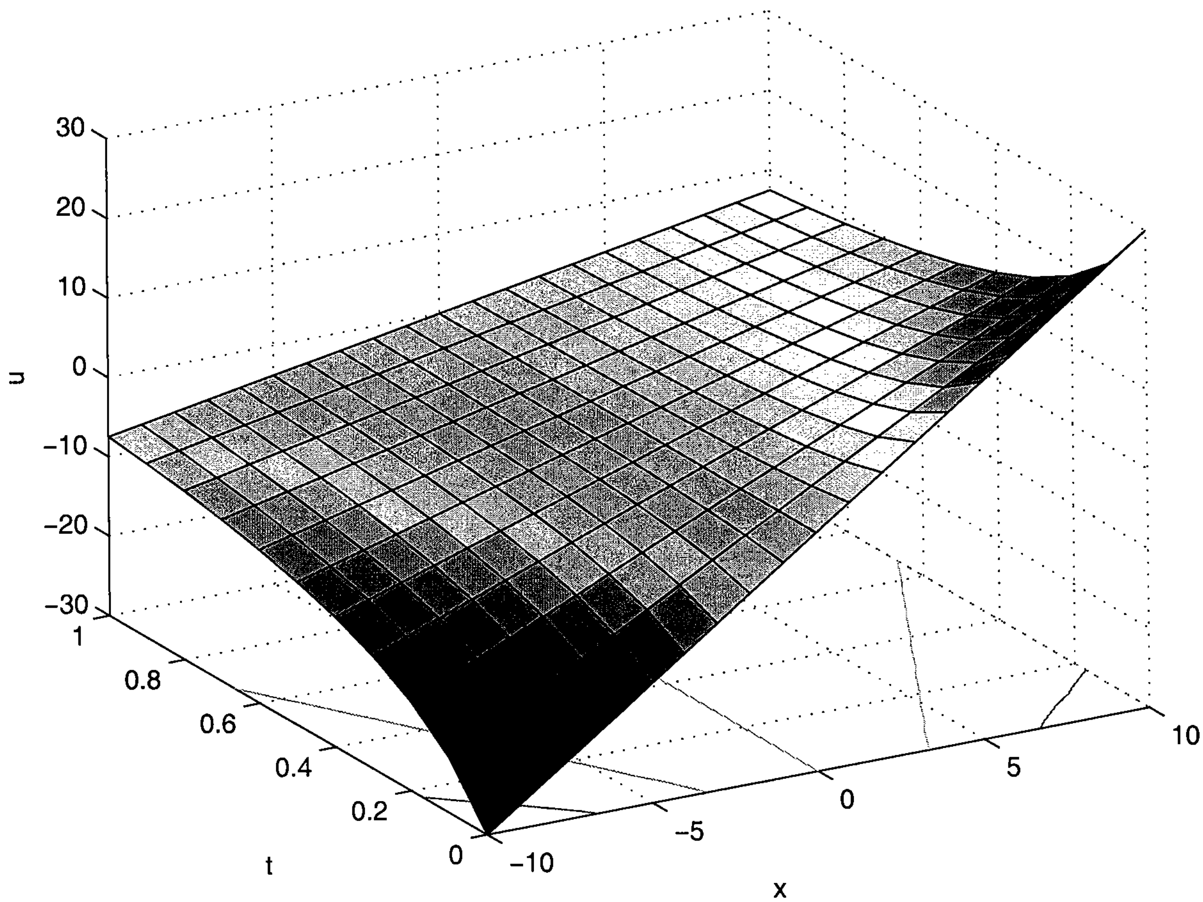
$u(x,t)$ is constant along a characteristic

② Suppose $\phi(x) = e^{-x^2/2}$

then we have to invert

$$u(x,t) = e^{-(x - u(x,t)t)^2/2}$$

to find $u(x,t)$ → difficult.

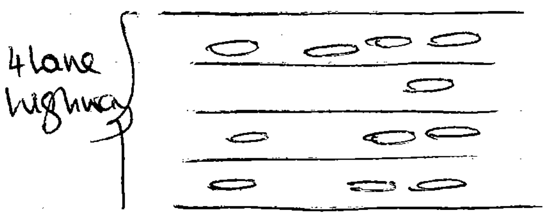


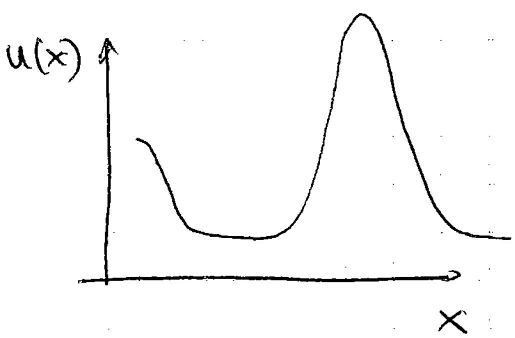
30
20
10
0
−10
−20
−30
u
1
0.8
0.6
0.4
0.2
0
t
−10
−5
0
5
10
x

2.4.4 Traffic flow

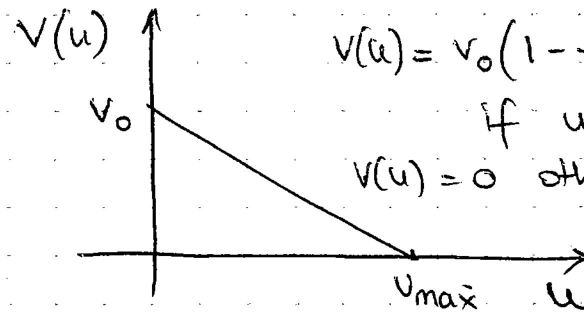
The study of traffic flow is an attempt at modelling (for example) the flow of cars on a road/highway, but also for example of information on a network, etc..

<u>Idea</u>: ① Model the road/network as a 1D line, with some density $u(x,t)$ of traffic (cars/information packets) at time t, position x..





② Model the velocity of the traffic flow as a function of the traffic density:



$V(u) = V_0\left(1 - \frac{u}{U_{max}}\right)$ if $u \in [0, U_{max}]$

$V(u) = 0$ otherwise

→ Flowing traffic has optimal velocity V_0 when u is small, and stalls when $u > U_{max}$

The conservation law for the car density $u(x,t)$ is simply

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u\,V(u)\right) = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[V_0 u\left(1 - \frac{u}{U_{max}}\right)\right] = 0$$

So here we have a conservation law with

$$F(u) = V_0 u\left(1 - \frac{u}{U_{max}}\right) \quad \text{if } 0 < u < U_{max}$$

$$= 0 \quad \text{otherwise}$$

$$\Rightarrow \quad F'(u) = v_0\left(1 - \frac{u}{u_{max}}\right) - \frac{v_0 u}{u_{max}}$$

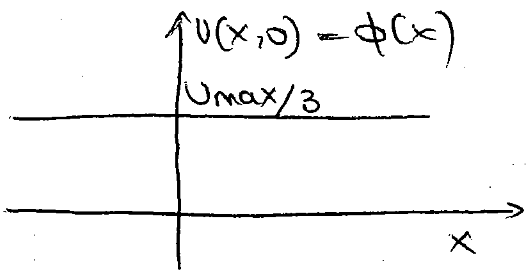
$$F'(u) \begin{cases} = v_0\left(1 - \dfrac{2u}{u_{max}}\right) & \text{if} \quad u \in [0, u_{max}] \\ = 0 & \text{otherwise} \end{cases}$$

- The solution to any initial traffic condition $u(x,0) = \phi(x)$ is given by the algebraic equation

$$u(x,t) = \phi\left(x - F'(u)t\right).$$

- The solution $u(x,t)$ is constant along characteristics, which are straight lines with slope $\dfrac{1}{F'(\phi(s))}$

<u>Example 1</u> Suppose we start with a uniform density of cars $u(x,0) = \dfrac{u_{max}}{3} \quad \forall x$.



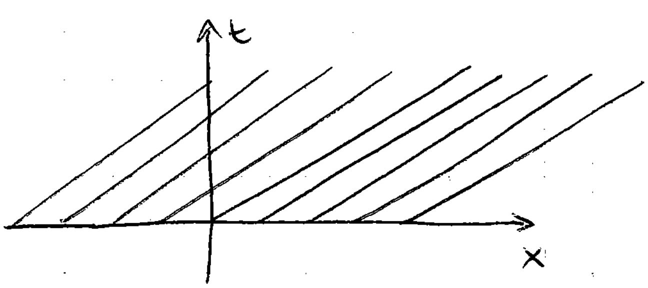
so $\phi(s) = \dfrac{u_{max}}{3} \quad \forall s$.

The characteristics are straight lines with equation

$$x = F'(\phi(s))t + s$$

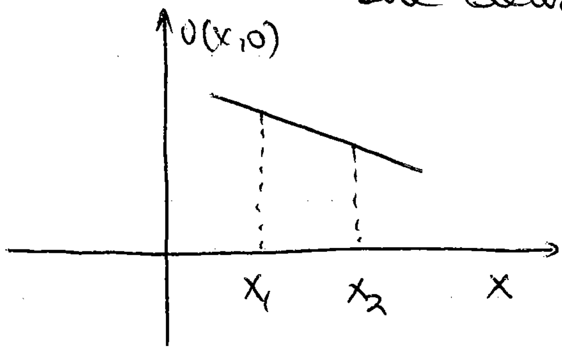
$$\Leftrightarrow \quad x = F'\left(\frac{u_{max}}{3}\right)t + s$$

$$\Leftrightarrow \quad x = \frac{v_0}{3}t + s$$

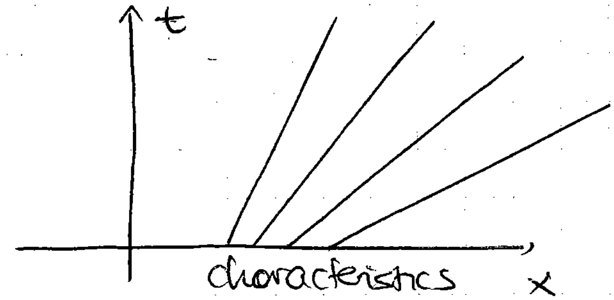


$\rightarrow$ since u is constant along a characteristic, we see that traffic is smoothly flowing at velocity $\dfrac{v_0}{3}$ and $u(x,t) = \dfrac{u_{max}}{3}$ is always constant

Example 2 Suppose there is a local decrease in the density of cars with x $\left(u(x,0) < \frac{u_{max}}{2} \right)$



$\Rightarrow$



$u(x_2,0) < u(x_1,0)$

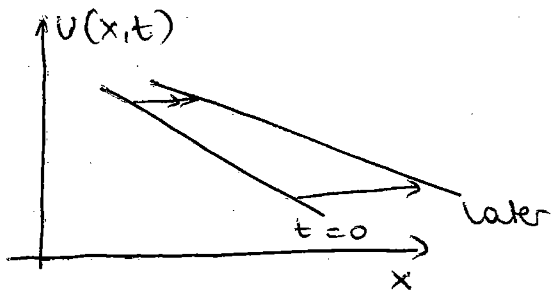
then $\phi(s_2) < \phi(s_1)$

$\Rightarrow \quad F'(s_2) > F'(s_1)$

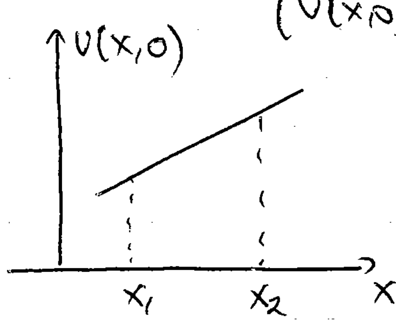
$\Rightarrow \quad \dfrac{1}{F'(s_2)} < \dfrac{1}{F'(s_1)}$

$\rightarrow$ the slope of the characteristics in the (x,t) plane decreases

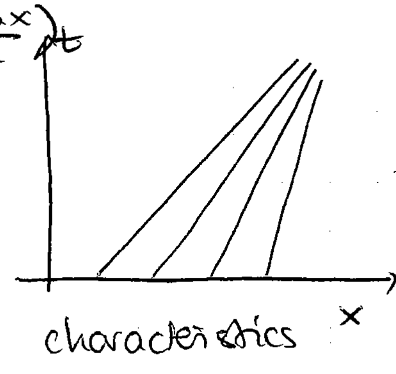
regions of less dense traffic move faster



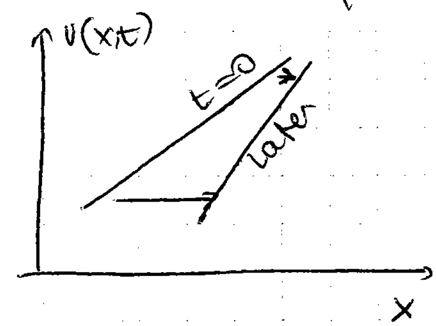
Example 3 Suppose there is a local increase in traffic $\left(u(x,0) < \frac{u_{max}}{2} \right)$



$\rightarrow$

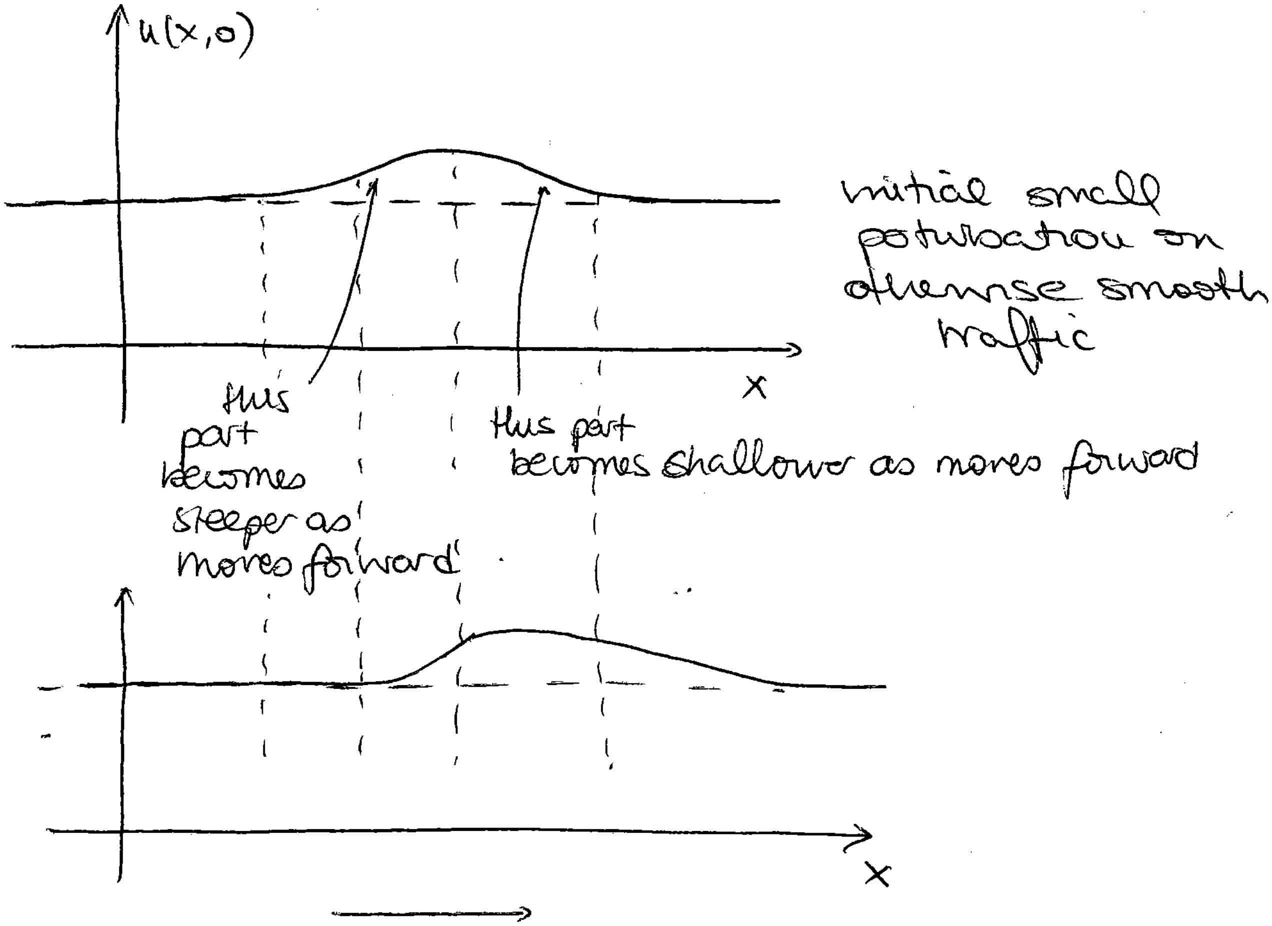


$\rightarrow$



$\Rightarrow$ This behavior leads to the emergence of <u>traffic waves</u> spontaneously.

The traffic wave moves forward or backward depending on the density of traffic compared with $\frac{u_{max}}{2}$



overall motion of perturbation.

<u>Homework</u> What happens when $u > \frac{U_{max}}{2}$?