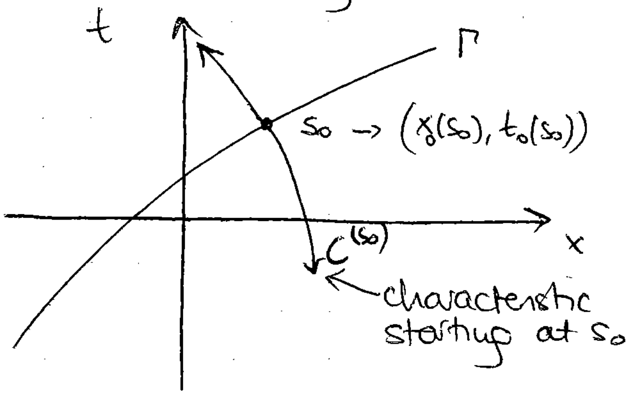
Step 2: Now that we have parametrized the "initial" condition curve we want to identify special curves along which the PDE behaves as an ODE. These are called characteristics.

① Suppose that for a selected point on the initial curve there exists only one characteristic emanating from it



$\Rightarrow$ let's parametrize this characteristic with the new parameter τ

$$C^{(s_0)} = \begin{cases} x^{(s_0)}(\tau) \\ t^{(s_0)}(\tau) \end{cases}$$

$\Rightarrow$ on this curve

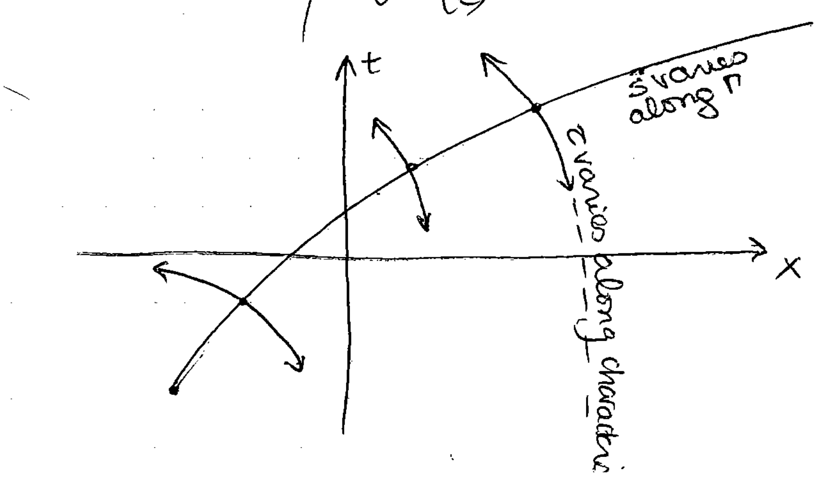
$$u\left(x^{(s_0)}(\tau), t^{(s_0)}(\tau)\right) = u^{(s_0)}(\tau)$$

u depends on τ only

② Now let's do this same construction for every point $[x_0(s), t_0(s)]$ on the initial condition curve

$\Rightarrow$ We get a family of characteristics, each starting from a point identified with the parameter s, and each parametrized with τ:

$$C^{(s)} \begin{cases} x^{(s)}(\tau) \\ t^{(s)}(\tau) \end{cases} \quad \text{with } u^{(s)}(\tau) = u\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$



③ Note that what we have really done, is to <u>remap</u> the (x,t) space onto the (s,τ) space

So that the function

$$u(x,t) \text{ is also } u\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$
$$= u(s,\tau)$$

with the added requirement that the PDE

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} \quad \text{behaves like an ODE}$$

when restricted to a characteristic curve ($s=$ constant)

<u>How do we do it?</u>

Note that $\left.\frac{\partial u}{\partial \tau}\right)_s$ is the derivative of u <u>along</u> a characteristic. (i.e holding s constant)

derivative of u w.r.t parameter τ at constant s

By multivariate chain rule, and using $\begin{pmatrix} x^{(s)}(\tau) \\ t^{(s)}(\tau) \end{pmatrix}$ on characteristic

$$\left.\frac{\partial u}{\partial \tau}\right)_s = \frac{\partial u}{\partial x}\left.\frac{\partial x}{\partial \tau}\right)_s + \frac{\partial u}{\partial t}\left.\frac{\partial t}{\partial \tau}\right)_s$$
$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial \tau}\left[x^{(s)}\right] + \frac{\partial u}{\partial t}\frac{\partial t^{(s)}}{\partial \tau}$$

Going back to the original PDE, if

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = \frac{\partial}{\partial \tau}\left[t^{(s)}\right]\frac{\partial u}{\partial t} + \frac{\partial}{\partial \tau}\left[x^{(s)}\right]\frac{\partial u}{\partial x}$$

then $\Rightarrow \left.\frac{\partial u}{\partial \tau}\right)_s = c_1 u + c_0$

Now the PDE looks like an ODE for s held constant, i.e, along a characteristic.

This occurs when $\frac{\partial t^{(s)}}{\partial \tau} = a \qquad \frac{\partial x^{(s)}}{\partial \tau} = b$

$$\Rightarrow \quad C^{(s)} = \begin{cases} t^{(s)} = a\tau + \text{constant specific to this characteristic} \\ x^{(s)} = b\tau + \text{constant} \quad '' \quad '' \quad '' \quad '' \end{cases}$$

Suppose we require that when $\tau = 0$ we are on the initial condition curve there at $\tau = 0$

$$t^{(s)} = t_0(s)$$
$$x^{(s)} = x_0(s)$$

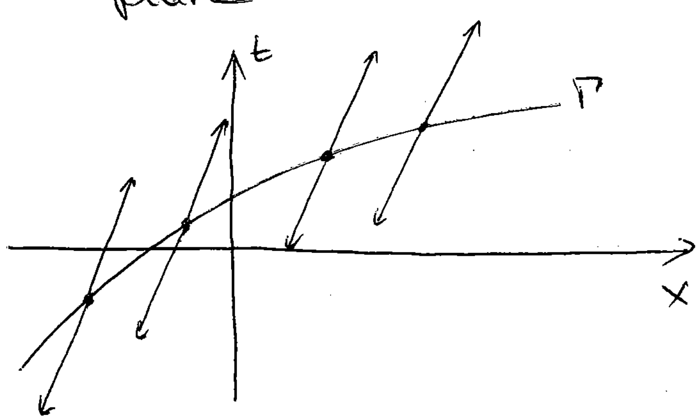
thus

$$C^{(s)} = \begin{cases} t^{(s)} = a\tau + t_0(s) \\ x^{(s)} = b\tau + x_0(s) \end{cases}$$

is the parametric equation for the characteristic emanating from the point $[x_0(s), t_0(s)]$ on the initial condition curve.

What do these characteristics look like?

They are straight lines with slope $\frac{a}{b}$ in the (x,t) plane



Note: this is only true when the PDE has constant coefficients (see later)

What is the solution to the PDE?

Now we have to solve $\frac{\partial u}{\partial \tau} = C_1 u + C_0$ for each s

$$\Rightarrow \quad u = A(s)\, e^{C_1 \tau} - \frac{C_0}{C_1} + \frac{C_0}{C_1} e^{C_1 \tau} \qquad \text{(check this)}$$

where the arbitrary constant $A(s)$ is chosen such that $u = U_0(s)$ when $\tau = 0$:

$$u(s,\tau) = U_0(s)\, e^{C_1 \tau} - \frac{C_0}{C_1} + \frac{C_0}{C_1} e^{C_1 \tau}$$

How do we get a solution in terms of (x,t)?

Invert the system (if possible)

$$\begin{cases} t = a\tau + t_0(s) \\ x = b\tau + x_0(s) \end{cases}$$

to write τ and s in terms of x and t, then plug into $u(s,\tau)$

Example 1 Suppose we want to solve the simple transport equation

$$\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial x} = 0$$

$$u(x,0) = e^{-x^2/2} \qquad \text{(a Gaussian)}$$

Step 1: Parametrize the initial condition

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2/2} \end{cases}$$

Step 2 The characteristic curves are such that

$\begin{pmatrix} a = 1 \\ b = V_0 \end{pmatrix}$

$$\begin{cases} \dfrac{\partial t}{\partial \tau} = 1 \\ \dfrac{\partial x}{\partial \tau} = V_0 \end{cases} \Rightarrow \begin{cases} t = \tau + t_0(s) \\ x = V_0\tau + x_0(s) \end{cases} \Rightarrow \begin{cases} t = \tau \\ x = V_0\tau + s \end{cases}$$

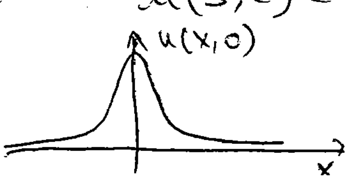
Step 3: The solution to $\dfrac{\partial u}{\partial \tau} = 0 \qquad (c_1 = c_0 = 0)$

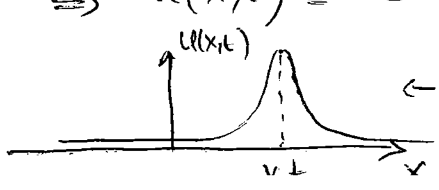
is $\quad u =$ constant on a characteristic

$$\Rightarrow \quad u = u_0(s) = e^{-s^2/2}$$

Step 4: $\begin{cases} t = \tau \\ x = V_0\tau + s \end{cases} \Rightarrow \begin{cases} \tau = t \\ s = x - V_0 t \end{cases}$

so $\quad u(s,\tau) = e^{-s^2/2} \Rightarrow u(x,t) = e^{-(x-V_0t)^2/2}$





← a "travelling" Gaussian i.e. horizontally moved.

Step 5: Always check the answer

$$\frac{\partial u}{\partial t} = -V_0(x-V_0t)e^{-(x-V_0t)^2/2}$$

$$\frac{\partial u}{\partial x} = (x-V_0t)e^{-(x-V_0t)^2/2}$$

so $\frac{\partial u}{\partial t} + V_0\frac{\partial u}{\partial x} = 0$ as required.

Example 2: (homework). Find the solution when $\begin{cases} a=0 \\ b=V_0 \\ c_i=-1 \\ c_0=0 \end{cases}$

Example 3: Let's consider the 'semilinear equation'

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$$

Q1: In which ways are the steps outlined above different

Q2: What is the solution to

$$\begin{cases} \frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2/2} \end{cases}$$

Answer:
- Obviously, the parametrization of the initial condition curve is unaffected by changes in the PDE
- If we assume there is still one characteristic emanating from each point on the curve then we can parametrize it with $C^{(s)} = \begin{cases} x^{(s)}(z) \\ t^{(s)}(z) \end{cases}$ as before
- This time, we want

$$\left.\frac{\partial u}{\partial z}\right)_s = a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t)$$

$$= \left(\frac{\partial t}{\partial z}\right)_s\frac{\partial u}{\partial t} + \left(\frac{\partial x}{\partial z}\right)_s\frac{\partial u}{\partial x}$$

$\Rightarrow$ Choose the characteristics such that

$$\begin{cases} \frac{\partial t^{(s)}}{\partial \tau} = a(x,t) \\ \frac{\partial x^{(s)}}{\partial \tau} = b(x,t) \end{cases}$$

On the characteristics, solve for u as

$$\frac{\partial u^{(s)}}{\partial \tau} = c(x,t,u)$$

then as before, get

$$\begin{cases} t = t(s,\tau) \\ x = x(s,\tau) \end{cases}, \text{ invert it to get } \begin{cases} \tau = \tau(x,t) \\ s = s(x,t) \end{cases}$$

and plug into the solution $u(s,\tau)$ to get $u(x,t)$.

$\Rightarrow$ $$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \qquad \begin{cases} a(x,t) = 1 \\ b(x,t) = x \\ c(x,t,u) = 0 \end{cases}$$

so we solve

$$\begin{cases} \frac{\partial t^{(s)}}{\partial \tau} = 1 & \rightarrow \quad t^{(s)} = \tau + \text{a function of } s \\ \frac{\partial x^{(s)}}{\partial \tau} = x & \rightarrow \quad \frac{\partial x}{x} = \partial \tau \\ \frac{\partial u^{(s)}}{\partial \tau} = 0 & \quad \hookrightarrow \ln x = \tau + \text{a function of } s \\ & \quad \hookrightarrow x^{(s)} = c(s)e^{\tau} \end{cases}$$

To satisfy the initial condition on the initial condition curve

$$\begin{cases} t^{(s)}(\tau=0) = t_0(s) = 0 \\ x^{(s)}(\tau=0) = x_0(s) = s \end{cases}$$

so

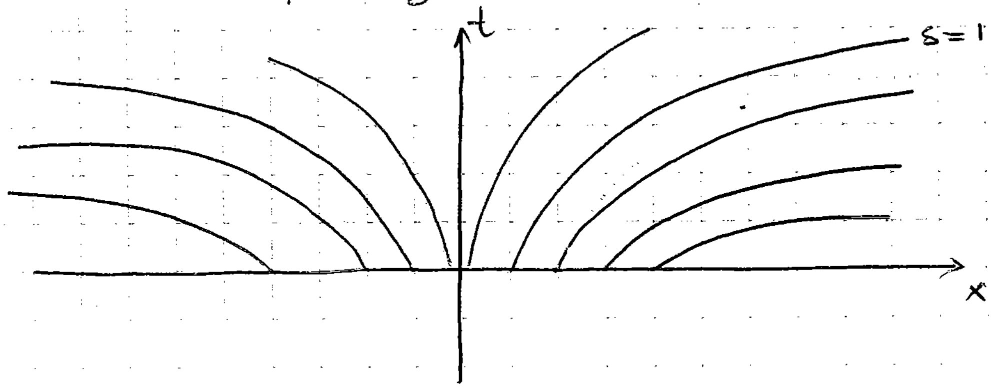
$$\begin{cases} t^{(s)} = \tau + t_0(s) = \tau \\ x^{(s)} = x_0(s)e^{\tau} = se^{\tau} \end{cases}$$

What do the characteristics look like?

Eliminating τ yields $x = se^{t}$ or

$$t = \ln\left(\frac{x}{s}\right) = \ln(x) - \ln(s)$$

$\Rightarrow$ a family of curves that are $\ln x$ - a constant



The solution for u is obtained by solving

$$\frac{\partial u^{(s)}}{\partial \tau} = 0 \quad \text{on each characteristic}$$

so u = a function of s only

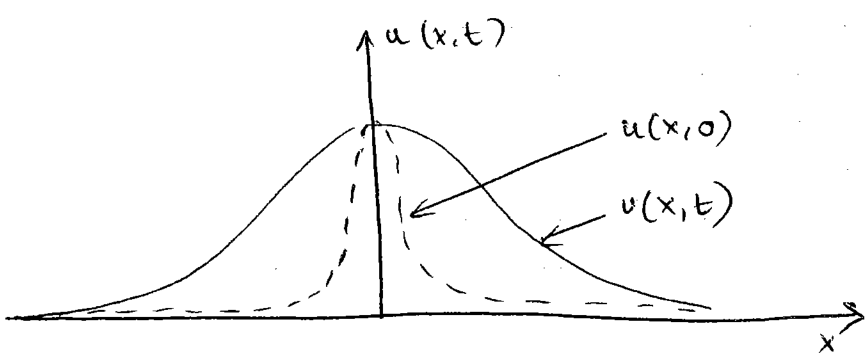
$$u = u_0(s) = e^{-s^2/2}$$

So finally, $u(s,\tau) = e^{-s^2/2}$

but $s = xe^{-t}$ so $\Rightarrow$ $u(x,t) = e^{-(xe^{-t})^2/2}$

$$= e^{-x^2/2e^{2t}}$$

This time, the Gaussian is <u>stretched</u>



What did we learn?

(1) Method of solution of linear, first order PDES

Step 1: Parametrize the initial condition curve

Step 2: If $a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t)$

then the characteristics are found by solving the system

$$\begin{cases} \frac{\partial t^{(s)}}{\partial \tau} = a(x,t) \\ \frac{\partial x^{(s)}}{\partial \tau} = b(x,t) \end{cases}$$

with the initial condition

$$t^{(s)}(\tau=0) = t_0(s)$$
$$x^{(s)}(\tau=0) = x_0(s)$$

Step 3: The solution to the PDE in (s,τ) is found by solving

$$\frac{\partial u^{(s)}}{\partial \tau} = c(x,t)$$

(note that x and t depend on s and τ)

Step 4: If possible, invert the system

$$\begin{cases} t(s,\tau) \\ x(s,\tau) \end{cases} \quad \text{to get} \quad \begin{cases} \tau(x,t) \\ s(x,t) \end{cases}$$

and plug into $u(s,\tau)$ to get $u(x,t)$.

Step 5 Check answer.

② Note.

- When the linear PDE is homogeneous ($c(x,t) = 0$) then

$$\frac{\partial u}{\partial \tau} = 0 \Rightarrow$$ u is constant along characteristics; in other words, the characteristics are contour levels of the solution $u(x,t)$.

- When the PDE is not homogeneous then u is not constant along characteristics. The characteristics propagate the initial condition according to the equation

$$\frac{\partial u^{(s)}}{\partial \tau} = c\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$

(see examples later)

③ Question

- What is the condition for the mapping

$$\begin{cases} \underline{x}(s,\tau) \\ t(s,\tau) \end{cases}$$ to be invertible?

- What happens if the characteristics are somewhere parallel to the initial condition curve?