MATH2004A – Test 1 – 4:35 pm - 5:25 pm, Oct 1

Name: Student Number:

Total: 20 marks
Closed book, no calculator!

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1. [8 points = 3 + 5] Let

$$f(x) = \begin{cases} 0, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in [0, \pi). \end{cases},$$

and let $f(x)$ be 2π-periodic. Find the Fourier coefficients a_0, a_n,

Solution:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} 1dx\right) = 1, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cos(nx)dx + \int_{0}^{\pi} 1\cos(nx)dx\right) = \frac{1}{n\pi}\sin(nx)|_0^\pi = 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\sin(nx)dx + \int_{0}^{\pi} 1\sin(nx)dx\right) = -\frac{1}{n\pi}\cos(nx)|_0^\pi \\ &= \frac{1}{n\pi}\left(1-(-1)^n\right) = \begin{cases} \frac{2}{n\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases} \end{aligned}$$

2. (12 points) Let $f(x) = \begin{cases} 1-x, & 0 \le x < 1. \\ 0, & 1 \le x < 2; \end{cases}$.

(i) (1.5 points) Let $f_{odd}(x)$ be the 4-periodic **odd** extension of $f(x)$. Find the expression of $f_{odd}(x)$ when $-1 < x < 0$.

Solution: Solution:

$$f_{odd}(x) = -f(-x) = -(1-(-x)) = -(1+x) = -1-x.$$

(ii) (1.5 points) Let $f_{even}(x)$ be the 4-periodic **even** extension of $f(x)$. Find the expression of $f_{even}(x)$ when $-1 < x < 0$.

Solution: Solution:

$$f_{even}(x) = f(-x) = (1-(-x)) = 1+x.$$

(iii) (4 points) Find the Fourier sine series.

Solution: Solution: Fourier sine series: for $n = 1, 2, 3, \ldots,$

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\sin(\frac{n\pi x}{2})\,dx \\ &= \int_0^1 (1-x)\sin(\frac{n\pi x}{2})\,dx \\ &= \left[-\frac{2}{n\pi}(1-x)\cos(\frac{n\pi x}{2}) - \frac{4}{n^2\pi^2}\sin(\frac{n\pi x}{2})\right]_0^1 \\ &= \frac{2}{n\pi} - \frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2}). \qquad (\text{ 3 points, 1 point for the formula}) \end{aligned}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi} - \frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2})\right\}\sin(\frac{n\pi x}{2}). \qquad (1point)$$

(iv) (4 points) Find a_0 and a_n of the Fourier cosine series.

Solution: Solution:

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{2}\int_0^2 f(x)\,dx \\ &= \int_0^1 (1-x)\,dx = \frac{1}{2}. \qquad (1point) \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\cos(\frac{n\pi x}{2})\,dx \\ &= \int_0^1 (1-x)\cos(\frac{n\pi x}{2})\,dx \\ &= \left[\frac{2}{n\pi}(1-x)\sin(\frac{n\pi x}{2}) - \frac{4}{n^2\pi^2}\cos(\frac{n\pi x}{2})\right]_0^1 \\ &= \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos(\frac{n\pi}{2}). \qquad (\text{ 3 points, 1 point for the formula}) \end{aligned}$$

(v) (1 point) Assume that the Fourier sine series in (ii) converges to B when $x = 11.7$. Find B.

Solution: Solution:

$$B = \frac{f_{odd}(11.7+) + f_{odd}(11.7-)}{2} = f_{odd}(11.7) = f_{odd}(11.7-12) = f_{odd}(-0.3) = -1+0.3 = -0.7.$$