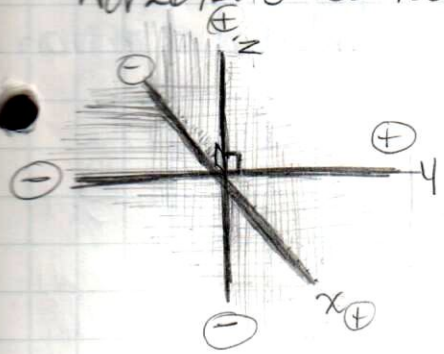
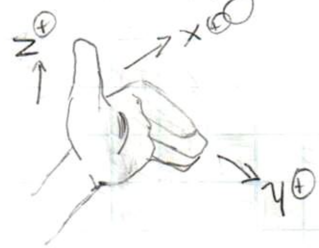
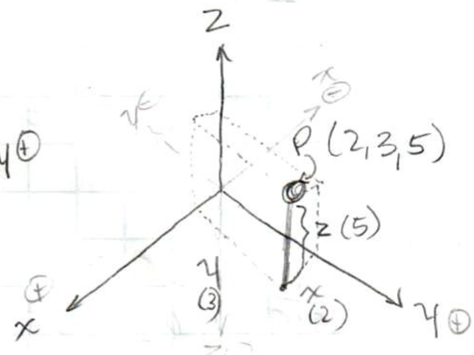
Nov 26/2013 3D vectors – MAT1339



z axis is a 3rd component to the cartesian plane representing a projection forward & backward from the plane. It is difficult to draw this on paper so a bit of imagination is necessary :)



Ⓡ hand



Divided into 8 octants:

x	y	z	
+	+	+	1
+	+	−	2
+	−	+	3
+	−	−	4
−	+	+	5
−	+	−	6
−	−	+	7
−	−	−	8

p(a, b, c)
(x, y, z)

if P is a vector then $\vec{P}[a,b,c]$

$|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$

the unity vector of z axis is $\boxed{K}$

(i, j, K) $i[1,0,0]$ $j[0,1,0]$ $K[0,0,1]$

they have similar properties in 3 dimensions:

$\vec{u}[a_1 b_1 c_1]$
$\vec{v}[a_2 b_2 c_2]$

summation $= \vec{u} + \vec{v}$ $[a_1 + a_2, b_1 + b_2, c_1 + c_2]$

multiplication with a scalar: $\vec{u} \cdot \lambda = [\lambda a, \lambda b, \lambda c]$ $\lambda \in \mathbb{R}$

decomposition into the unity vector:
$\vec{u} = [ai + bj + cK]$

dot product: $\vec{u} \cdot \vec{v} = a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$

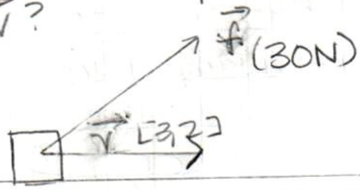
$proj_{\vec{v}} \vec{u} = \left| \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \right|$ ← absolute value only for scalar proj. because it is a scalar length

$\overrightarrow{proj_{\vec{v}} \vec{u}} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$ ← if this is ⊖ this means we have a vector in the opposite direction of $\vec{v}$

$\cos \angle \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

Application of dot product

- work done by force $\vec{f}$ exerted on an object & moves it in the direction of $\vec{w}$?



moved from [0,2] to [2,7]

what is the work done by force $\vec{f}$?

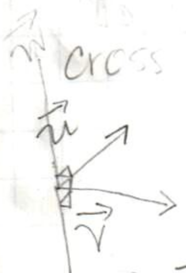
direction of $\vec{f} = \dfrac{\vec{w}}{|\vec{w}|} \Rightarrow \vec{f} = 30\dfrac{\vec{w}}{|\vec{w}|} = 30\left[\dfrac{3}{\sqrt{13}}, \dfrac{2}{\sqrt{13}}\right] = \left[\dfrac{90}{\sqrt{13}}, \dfrac{60}{\sqrt{13}}\right]$

$\hookrightarrow \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$

$\vec{v} = [2,7]-[0,2] = [2,5]$

$\vec{W} = \vec{f}\cdot\vec{v} = \left[\dfrac{90}{\sqrt{13}}, \dfrac{60}{\sqrt{13}}\right]\cdot[2,5] = \dfrac{180+300}{\sqrt{13}} = \dfrac{480}{\sqrt{13}}$ joules

cross product



$\vec{u}\times\vec{v}$ = a vector which is orthogonal to both $\vec{u}$ & $\vec{v}$

$\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\angle\theta$

$\vec{u}[a_1, a_2, a_3]$

$\vec{v}[b_1, b_2, b_3]$

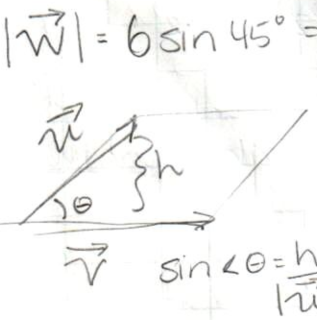
$$\begin{array}{cccccc} i & j & k & i & j & k \\ a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_1 & b_2 & b_3 \end{array}$$

?

$\vec{u}\times\vec{v} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_2b_1 - a_1b_2]$

Ex: $|\vec{u}| = 3$ $|\vec{v}| = 2$ $\theta = 45°$

$|\vec{w}| = 6\sin 45° = 6\dfrac{\sqrt{2}}{2} = 3\sqrt{2}$



area $= |\vec{v}|\cdot h = |\vec{v}||\vec{u}||\sin\theta|$

$\sin\angle\theta = \dfrac{h}{|\vec{u}|}$

Properties of cross product:

① $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

② $\angle\theta = 0 \Rightarrow \vec{u}\times(\lambda\vec{u}) = \vec{0}$

③ $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\times\vec{w}$

④ $\lambda(\vec{u}\times\vec{v}) = \lambda\vec{u}\cdot\vec{v} \quad \lambda\vec{v}\times\vec{u}$

⑤ $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})\times(\vec{u}+\vec{w})$

based on coordinates:

Ex: $\vec{u}\times\vec{v}$

$\vec{u}[1,2,3]$

$\vec{v}[-3,1,0]$

$$\begin{array}{cccccc} i & j & k & i & j & k \\ 1 & 2 & 3 & 1 & 2 & 3 \\ -3 & 1 & 0 & -3 & 1 & 0 \end{array}$$

$\vec{u}\times\vec{v} = [(2\cdot 0)-(3\cdot 1), (3\cdot -3)-(1\cdot 0), (1\cdot 1)-(2\cdot -3)]$

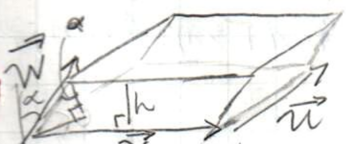
$= [(0-3), (-9-0), (1+6)]$

$= [-3, -9, 7]$

the area of the parallelogram based on $\vec{u}$ & $\vec{v}$

$= \sqrt{-3^2+-9^2+7^2} = \sqrt{9+81+49} = \sqrt{139}$

cross dot product



parallelpiped

(each side is a parallelogram)

what is the volume?

$=$ area of the bottom $\times$ height

$(\vec{u}\times\vec{v})$ $\quad$ $(\cos\alpha\cdot|\vec{w}|)$

$= |(\vec{u}\times\vec{v})\cdot\cos\alpha|\vec{w}||$

$$\begin{array}{cccccc} i & j & k & i & j & k \\ 1 & 0 & 2 & 1 & 0 & 2 \\ -1 & 1 & 3 & -1 & 1 & 3 \end{array}$$

Eg: $\vec{u} = [1,0,2]$

$\vec{v} = [-1,1,3]$

$\vec{w} = [0,3,5]$

$= ((0\cdot 3)-(2\cdot 1), (2\cdot -1)-(1,3), (1,1)-(0,-1))$

$= (0-2), (-2-3), (1+1)$

$= [-2, -5, 2]$

$|\vec{w}| = \sqrt{0^2+3^2+5^2} = \sqrt{9+25} = \sqrt{34}$

$|(\vec{u}\times\vec{v})\cdot\cos\alpha|\vec{w}|| = [-2,-5,2]\cdot\sqrt{34}\cos\alpha$

lines: 2D space { Scalar, Vector, parametric } equations

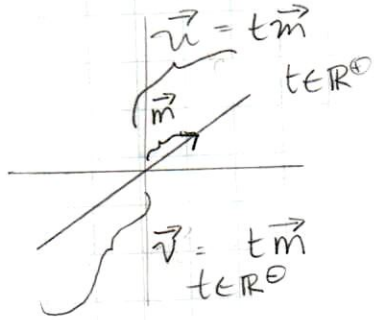
3D space { Vector, parametric }

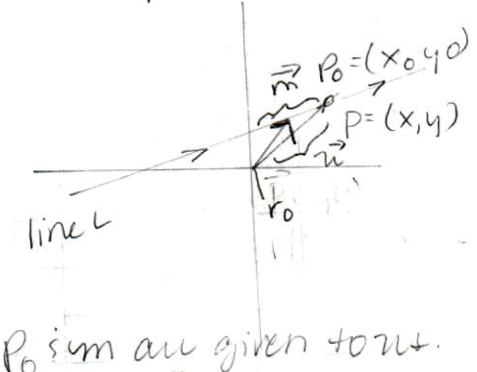
2D scalar $= m =$ slope

& one point on the line $= (x,y)$

$y - y_0 = m(x - x_0)$

a line is a geometric object that can be produced by a vector





P_0 & m are given to us.

$m = [a_0, b_0]$

$\vec{u} = [x, y]$ $\quad [x, y] = P_0 + t[a, b], t \in \mathbb{R}$

Eg: $\vec{m}$ = the direction vector of the line

$m = [1, 2]$

$P_0 = (3, 5)$ is on the line

what is the vector equation of the line?

$[x, y] = r_0 + t\vec{m} = [3, 5] + t[1, 2]$

$[x, y] = [3 + t, 5 + 2t]$

the position vector of an arbitrary point on the line

now all we need is any value for t & we can find $[x, y]$ for any point

$$[x, y] = [x_0, y_0] + t[a_0, b_0] = [x_0 + ta_0, y_0 + tb_0] \qquad \begin{cases} x = (x_0 + ta_0) \\ y = (y_0 + tb_0) \end{cases}$$

parametric equation for the line

t is the <u>parameter</u>

b) check if $A(1, 2) \in L$ or not

if $A \in L$ then it will be satisfied by the equation of the line

usually we use parametric equations

the t must = for $x + y$.

Eg $\begin{cases} x = 3 + t \\ y = 5 + 2t \end{cases}$

$1 = 3 + t \qquad t_1 = 1 - 3 \qquad t_1 \neq t_2 \therefore A \notin L.$

$2 = 5 + 2t \qquad t_2 = \frac{2 - 5}{2}$

c) what is the scalar equation of the line?

(from a vector or parametric equation)

-eliminate t between the two equations

$t = x - 3$

$y = 5 + 2(x - 3) = 5 + 2x - 6 = 2x - 1$ this is the scalar equation for lines in 3D space everything is the same b/c summation of vectors is the same in 2D & 3D space.

- $[x, y, z] = r_0 + t[a_0, b_0, c_0]$ vector equation in 3D space

$= [x_0, y_0, z_0] + t[a_0, b_0, c_0]$

$$\left.\begin{aligned} x &= x_0 + ta_0 \\ y &= y_0 + tb_0 \\ z &= z_0 + tc_0 \end{aligned}\right\}$$ parametric equation in 3D space

no scalar equation for the line in 3D space - $[x, y, z] \cdot$ slope in 3D space is a plane.

Eg: What is the equation of a line that passes through the two points $A = (2, -1, 5)$ & $B = (3, 6, -4)$

$\overrightarrow{AB}$ can be considered as the direction vector on the line. (or $\overrightarrow{BA}$) $\overrightarrow{BA} = (3-2, 6+1, -4-5) = (1, 7, -9)$

r_0 can be A or B

$[x, y, z] = r_0 + t[a_0, b_0, c_0]$

$[x, y, z] = (2, -1, 5) + t[1, 7, -9]$ Vector equation

- $$\left.\begin{aligned} x &= 2 + t \\ y &= -1 + 7t \\ z &= 5 - 9t \end{aligned}\right\}$$ parametric equation

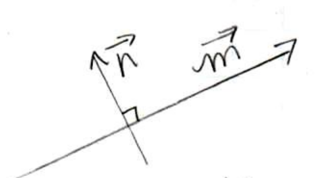
b) check if 0,0,0 is on the line

$0 = 2 + t \quad t = -2$

$0 = -1 + 7t \quad t = \frac{1}{7}$

> Since these 2 are not equal we don't need to check z to know $0,0,0 \notin$ the line.

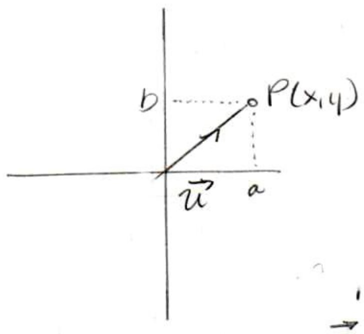
can also ask to check x, y, z intercepts $(x, 0, 0)$ $(0, y, 0)$ $(0, 0, z)$



$\vec{n}$ is orthogonal to the line & is called the normal vector of the line.

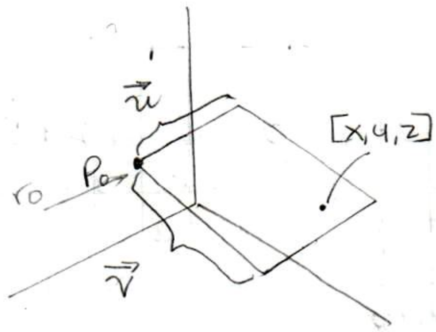
- If you have $n = [a, b]$ you can find the scalar equation.

$a_x + b_y = 0$



$u = a\vec{i} + b\vec{j}$

↳ a linear combination of 2 vectors



$z=0$ = x,y plane

$y=0$ = xz plane

$x=0$ = yz plane

2 vectors on the plane initiated from the same point

vector equation of the plane $[x,y,z] = \vec{r}_0 + t\vec{u} + s\vec{v}$

$t, s \in \mathbb{R}$

$\vec{n}$ is the normal vector of the plane: $\vec{n} \perp \vec{w}$, $\vec{w} \in$ plane

$\vec{n} \cdot \vec{w} = 0$

$\vec{n} = [a_0, b_0, c_0]$ $\quad \vec{w} = [x,y,z]$

$\vec{n} \cdot \vec{w} = a_0 x + b_0 y + c_0 z = 0$ ← scalar equation of the plane