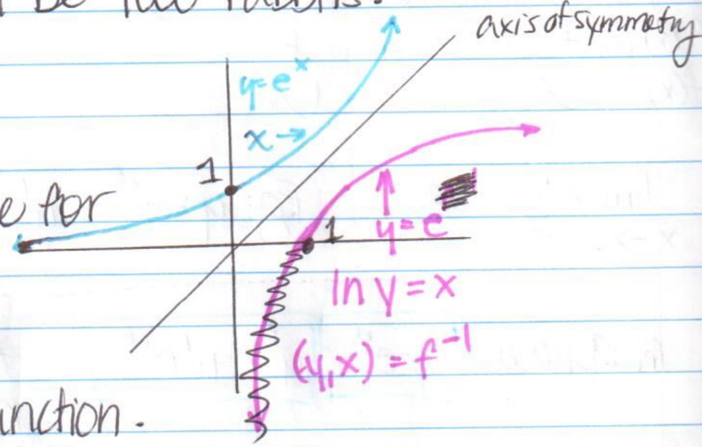
## Logarithms:

$\boxed{y=e^x \quad \log_e y = x}$ $\boxed{\log_e = \ln}$ so $\log_e y = x = \ln y = x$.

ln = natural logarithm or the log base e.

Eg: $R(x)=e^x$ $R(x)$ = # of rabbits $x$ = # of years.
How many years until there will be 1000 rabbits?




$1000 = e^x \Rightarrow \ln 1000 = x$

in this function we know the value for y, but we don't know x.

the blue line is the function $e^x$
the purple line is the reciprocal function.
The graph of $f^{-1}$ is a mirror image of the graph of x.
To sketch the graph of $f^{-1}$ just change $(x,y) \to (y,x)$ for all points.

$f(x)=e^x$ $x$ $-\infty$ $+\infty$ ] Dom $f(x)$
$f(x)$ ↗ ↗
$f'(x)$ $0^+$ + + $+\infty$ ) Image $(f(x))$ and vice versa.
$f(^{-1})$: $0^+$ + + $+\infty$ Dom $f(x)^{-1}$
$(f^{-1})'$: + +
$f^{-1}$: ↗ ↗

we call $f^{-1}$ the inverse function or the reciprocal function.
for this function instead of writing $f^{-1}$ we write log.
log is a name for $f^{-1}$ where f is an exponential map.

$f(x) = e^{(x)}$ $f^{-1}(x) = \log_e x$. Any x for $\log_e x$ will be y for $e^x$.
(the domain of $\log_e x$ is the image of $e^x$)
(the domain of $e^x$ is the image of $\log_e x$)

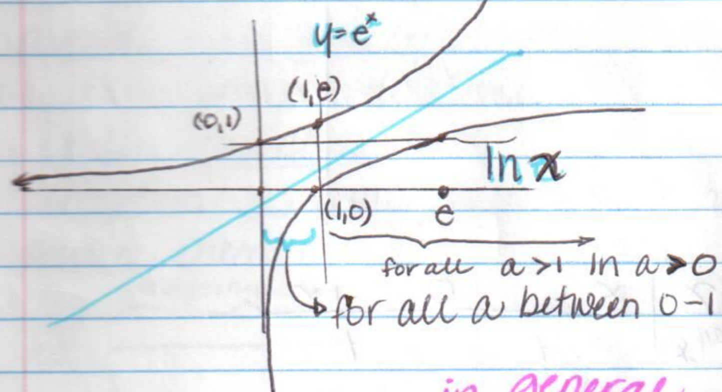
One of the useful properties of logarithms is linearization.
It can change a product into a summation:

$\log_e(ab) = \log_e a + \log_e b$
$\log_e(2\cdot 3) = \log_e 2 + \log_e 3$

Dom $\ln x = \{\mathbb{R}^+\} = \text{Im } e^x$

Dom $e^x = \mathbb{R} = \text{Im } \ln x$

this is why we define $a^x$ as $a>0$




in general

$\boxed{y=a^x \quad y'=a^x(\ln x)}$ $\quad y'=a^x x'(\ln x)$

Eg: $y=3^x \quad y'=3^x \ln 3$ $\quad \boxed{y=3^{(x^2+x)} \quad y'=3^{(x^2+x)}(2x+1)\ln 3}$

$y=3^{\sqrt{x}} \quad y'=3^{\sqrt{x}}\dfrac{1}{2\sqrt{x}}\ln 3$

$f(x)=a^x \Rightarrow f'(x)=\ln a \cdot a^x \Longrightarrow f''(x)=\ln a \cdot \ln a \cdot a^x = (\ln a)^2 a^x$

| $x$ | $-\infty$ | | $+\infty$ |
|---|---|---|---|
| $f'(x)$ | | $+$ | |
| $f(x)$ | $0^+$ | $\nearrow$ | $+\infty$ |

| $x$ | $-\infty$ | | $+\infty$ |
|---|---|---|---|
| $f''(x)$ | | $+$ | |
| $f(x)$ | | $\smile$ | |

$\boxed{\text{When } a>1 \Rightarrow \ln a>0 \Rightarrow f'>0}$ $\therefore$ the graph of $a^x$ looks like the graph of $e^x$ but the values of y are different

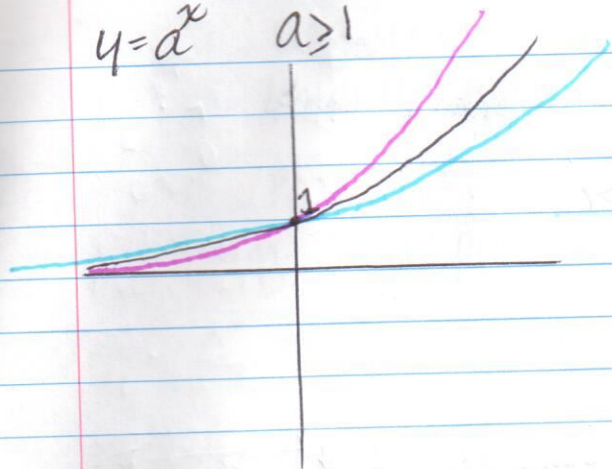
$\boxed{\text{When } 0<a<1 \Rightarrow \ln a<0 \therefore f'<0}$

This makes f(x) a decreasing function

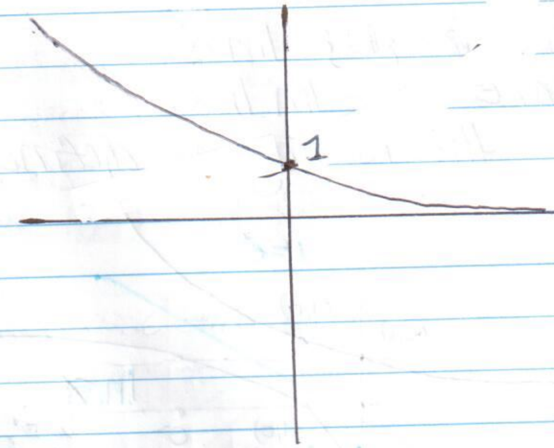
$\lim\limits_{x\to-\infty} a^x \; (0<a<1) = \dfrac{1}{a^x} = \dfrac{1}{0^+} = \infty^+$

$\lim\limits_{x\to+\infty} a^x \; (0<a<1) = \dfrac{1}{a^x} = \dfrac{1}{a^\infty} = \dfrac{1}{\infty} = 0^+$

$y=a^x \quad a \geq 1$




$y=a^x \quad 0<a<1$




$y=a^x \quad 0<a<1$

| $x$ | $-\infty$ | | $0$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+\infty$ | $-$ | $1$ | $-$ | $0^+$ |
| $f(x)$ | | $\searrow$ | | $\searrow$ | |

| $x$ | $-\infty$ | | $0$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f''(x)$ | | $+$ | | $+$ | |
| | | $\cup$ | | $\cup$ | |

$\ln a^n = \ln a \cdot \ln a \cdot \ln a \cdot \ldots$ n times

$$\boxed{\ln a^n = n \ln a}$$

Eg: $\ln e^8 = 8 \ln e$

$f(x)=y \qquad f^{-1}(f(x))=x$

$\left\{\begin{array}{l} f(x)=e^x \\ f^{-1}(x)=\ln x \end{array}\right\} \; f^{-1}(f(x)=\ln e^x = x$

$\left\{\begin{array}{l} f(x)=\ln x \\ f^{-1}(x)=e^x \end{array}\right\} \; f^{-1}(f(x)) = e^{f(x)} = e^{\ln x} = x$

## Chapter 6: Vectors

2 elementary notions in mathematics: scalars & vectors

a scalar is any real number.

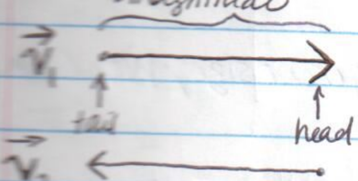
- for a scalar we have 1 identifier: the magnitude

  Eg: height, weight, temperature

a vector has 2 identifiers:

- the magnitude (size)
- the direction

Definition: a vector is any segment of a line with both a magnitude and a direction.




same size, different direction

$\vec{V}_1$ ← this arrow is important as it shows that $V_1$ is a vector & not scalar.

$|\vec{v}|$ = notation for "the size of the vector"

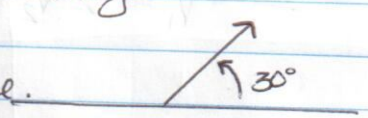
We can analyze vectors in 2 ways:

1: geometrically:

there are 3 ways to look at vectors geometrically:

a) horizontal axis:

put the tail on the origin & move counter clockwise.




b) true bearing:

put the tail on a vertical line representing North and move clockwise from the origin to the vector:

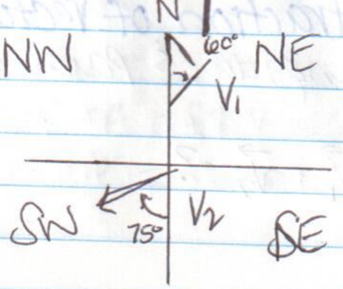
N60°

N120°


c) quadrant bearing

$V_1 = N60°E$

$V_2 = S75°W$



A, B, C, D: $A \swarrow\!\!\nearrow B$ ... If the vectors are parallel to each other we say they are parallel vectors. Being parallel doesn't tell us anything about the size or direction of the vectors.

Notation: $AB \parallel CD$

If two vectors have the same size & the same direction we say they are equivalent to each other.

$\vec{v}_1 \longrightarrow$ $\qquad \vec{v}_1 \sim \vec{v}_2 \qquad \vec{v}_3 \longrightarrow \qquad \vec{v}_3 \nsim \vec{v}_4$ ← not equivalent

$\vec{v}_2 \longrightarrow \qquad \qquad \vec{v}_4 \longleftarrow \qquad \vec{v}_3 \parallel \vec{v}_4$ & $|\vec{v}_3| = |\vec{v}_4|$

For 2 parallel vectors with the same direction and different size, we can compare their sizes to each other:

$\vec{v}_1 \longrightarrow \qquad |\vec{v}_1| = k|\vec{v}_2|$ } they are colinear.

$\vec{v}_2 \longrightarrow \qquad \vec{v}_1 \parallel \vec{v}_2$

K is a scalar. $k|\vec{v}_2|$ means k times $\vec{v}_2$.

instead of writing $\left\{ \begin{matrix} |\vec{v}_1| = k|\vec{v}_2| \\ \vec{v}_1 \parallel \vec{v}_2 \end{matrix} \right\}$ we can abbreviate it as $\vec{v}_1 = k\vec{v}_2$.

$\vec{v}_1 \rightarrow$

$\vec{v}_2 \longleftarrow$ $\qquad = \vec{v}_1 = -2\vec{v}_2$ ← parallel, double the size & in the opposite direction

K is a scalar that can also be <1:

$\vec{v}_1 = \frac{1}{2}\vec{v}_2 \qquad \longrightarrow \vec{v}_2 \qquad \longrightarrow \vec{v}_1$

$\vec{v}_1 = -\frac{1}{2}\vec{v}_2 \qquad \longrightarrow \vec{v}_2 \qquad \longleftarrow \vec{v}_1$

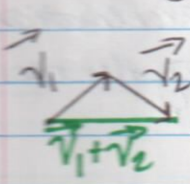
Some vectors are zero-directional but with no size. "The null vector" $\vec{0}$

## Addition & Subtraction of vectors:

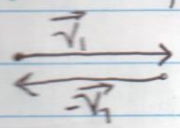
2 methods: Triangular & Parallelogram

$\vec{v}_1 \nearrow \qquad \searrow \vec{v}_2 \qquad \vec{v}_1 + \vec{v}_2 = ?$

① triangular method (tail on head.)



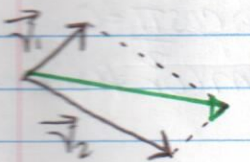

Rearrange the vectors to form a triangle & then draw a new vector representing the addition. The new vector is the resultant vector.




$\vec{v}_1 - \vec{v}_1 = \vec{v}_1 + (-\vec{v}_1) = \vec{0}$

do not forget signs & arrows – very important

② Parallelogram method (tail to tail)




Put the tail of $\vec{v}_1$ on the tail of $\vec{v}_2$, & draw a parallelogram representing these vectors. The diagonal of the parallelogram is the resultant vector.

Some properties of the addition of vectors:

1) $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$ (commutative)

2) $\vec{v}_1 + \vec{0} = \vec{v}_1$

3) $(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$ (associative property)

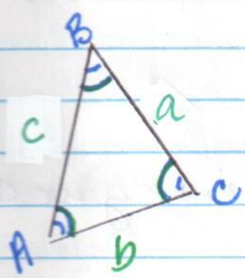
4) $k(\vec{v}_1 + \vec{v}_2) = k\vec{v}_1 + k\vec{v}_2$ (distribution property)

5) $k \,\&\, \ell \in \mathbb{R} \Rightarrow (k\ell)\vec{v}_1 = k(\ell\vec{v}_1)$

$$= \ell(k\vec{v}_1)$$
$$= \ell k(\vec{v}_1)$$

Eg: $3(2\vec{v} - 4\vec{w}) = 3(2\vec{v}) + 3(-4\vec{w}) = 6\vec{v} + (-12\vec{w}) = 6\vec{v} - 12\vec{w}$

Remember:



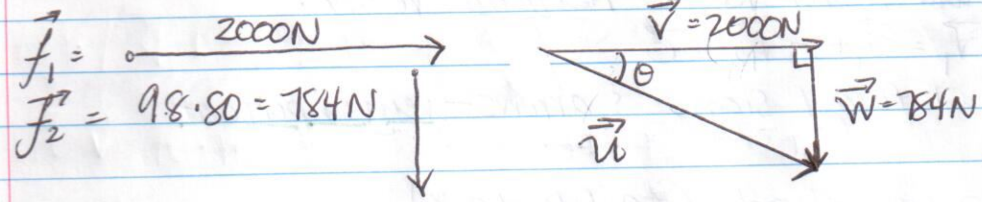

sin rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Eg: a clown with a mass of 80 kg is shot out of a cannon w/a horizontal velocity of 2000N. The vertical force is the acceleration due to gravity ($9.8 m/s^2 \circ 80kg$) multiplied by the mass of the clown. Find the magnitude & direction of the resultant vector.



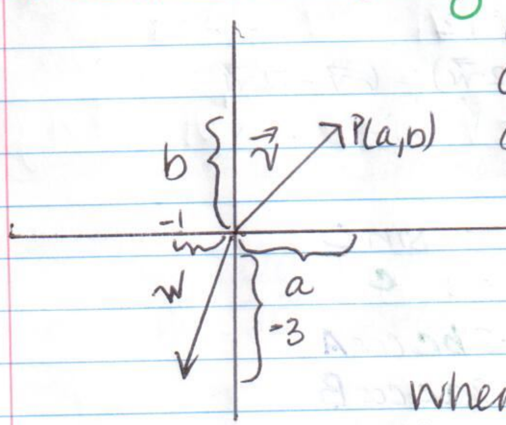

remember that when an angle is 90° that's $= \frac{\pi}{2}$ so $\cos\frac{\pi}{2} = 0$ we can use the cosine rule & eliminate the last term (or the pythagorean property)

$\vec{u}^2 = \vec{v}^2 + \vec{w}^2 \quad \vec{u} = \sqrt{\vec{v}^2 + \vec{w}^2} \quad \vec{u} = \sqrt{2000^2 + 784^2} = \sqrt{4614656} = 2148N$

Direction: $\frac{\sin\theta}{-\vec{w}} = \frac{\sin 90}{\vec{u}} \quad \sin\theta = \frac{1 \circ \vec{w}}{\vec{u}} = \frac{-784}{2148} \quad \sin^{-1}\left(\frac{-784}{2148}\right) = \theta$

we use −784 because gravity is ↓, clockwise from the horizon & the angle is <u>below</u> the horizon of the first vector.

## Cartesian vectors (analytical method)




consider a vector to begin at the origin of the cartesian plane. This means the head has coordinates in the plane.

$\vec{v}_1$ is pointing to point p w/ coordinates (a,b)

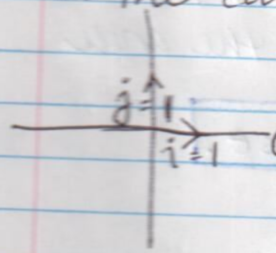
$\vec{v}_1 = [a, b]$

$\vec{w} = [-1, -3]$

When a vector is given in this way we call it a <u>position</u> <u>vector</u>.

## The unity vector

In the cartesian plane we have two important vectors:



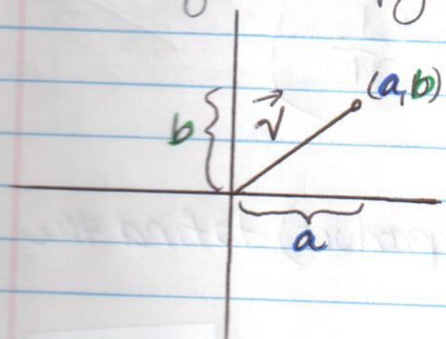

$i$ is a vector on the $x$ axis with length = 1.
$j$ is a vector on the $y$ axis with length = 1.

## Vector manipulatives:

addition: $\vec{v}_1 + \vec{v}_2 = [a,b] + [c,d] = [a+c, b+d]$

multiplication by a scalar: $K[a,b] = [Ka, Kb]$

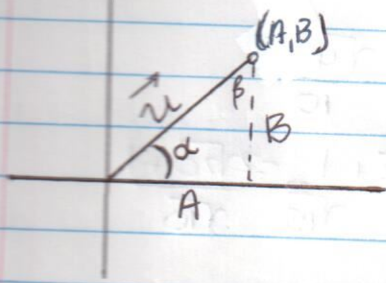
according to the pythagorean theorem, $|\vec{v}| = \sqrt{a^2+b^2}$




Eg: $\vec{v} = [3,4]$
$|\vec{v}| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Each vector can be expressed in terms of the unity vector: as multiples of the unity vector.

$\vec{v} = a\underset{(i)}{(1,0)} + b\underset{(j)}{(0,1)}$ $\quad \vec{v} = [a,0] + [0,b] = [ai + bj]$
$= [ai, bj]$




$\cos\alpha = \frac{A}{|\vec{u}|}$ $\quad \sin\alpha = \frac{B}{|\vec{u}|}$ $\Rightarrow A = |\vec{u}|\cos\alpha$
$B = |\vec{u}|\sin\alpha$

$\cos\beta = \frac{B}{|\vec{u}|}$ $\quad \sin\beta = \frac{A}{|\vec{u}|} \Rightarrow A = |\vec{u}|\sin\beta$
$B = |\vec{u}|\cos\beta$

Eg: $\vec{u} = [2,3]$ $\quad |\vec{u}| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

$\cos\angle\alpha = \frac{2}{\sqrt{13}}$ $\quad \alpha = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$

Eg: $\vec{v} = [-3,-1]$ $\quad |\vec{v}| = \sqrt{3^2+1^2} = \sqrt{10}$

$\cos\angle\alpha = \frac{2}{\sqrt{10}}$ $\quad \alpha = \cos^{-1}\left(\frac{2}{\sqrt{10}}\right)$

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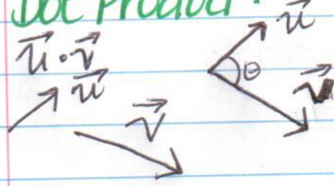
## Dot Product:

$\vec{u} \cdot \vec{v}$



consider an equivalent vector to $\vec{v}$ to be initiated from the tail of $\vec{u}$. $\theta$ is the angle between them.

The dot product is defined as $\boxed{\vec{u} \cdot \vec{v} = |\vec{u}| \times |\vec{v}| \times \cos\theta}$.

Eg: $|\vec{u}| = 10$ $|\vec{v}| = \sqrt{5}$ ; $\theta = 30°$

$\vec{u} \cdot \vec{v} = 10 \cdot \sqrt{5} \cdot \cos 30 = 10 \cdot \sqrt{5} \cdot \frac{\sqrt{3}}{2} = 5\sqrt{15}$

$\vec{u} = [a, b]$ $\quad \vec{u} \cdot \vec{v} = [a \cdot c + b \cdot d]$
$\vec{v} = [c, d]$ $\quad = [ac + bd]$
$\vec{u} = [\sqrt{2}, 1]$ $\quad \vec{u} \cdot \vec{v} = [2 \cdot \sqrt{2} + 1 \cdot \frac{1}{\sqrt{10}}] = -2\sqrt{2} + \frac{1}{\sqrt{10}}$
$\vec{v} = [-2, \frac{1}{\sqrt{10}}]$

the answers are all scalar

We can also use the dot product (aka scalar product) to find the angle between 2 vectors:

$|\vec{u}| \cdot |\vec{v}| \cdot \cos\theta = \vec{u} \cdot \vec{v}$ $\qquad \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$

Eg: $\vec{u} = [\sqrt{2}, 1]$ $\qquad |\vec{u}| = \sqrt{\sqrt{2}^2 + 1^2} = \sqrt{3}$
$\vec{v} = [-2, \frac{1}{\sqrt{10}}]$ $\qquad |\vec{v}| = \sqrt{-2^2 + \left(\frac{1}{\sqrt{10}}\right)^2} = \sqrt{4 + \frac{1}{10}} = \sqrt{\frac{41}{10}}$

$\left(\left(10^{-\frac{1}{2}}\right)^2 = 10^{-1}\right)$ $\qquad \vec{u} \cdot \vec{v} = [\sqrt{2} \cdot -2 + 1 \cdot \frac{1}{\sqrt{10}}] = -2\sqrt{2} + \frac{1}{\sqrt{10}} = \frac{-2\sqrt{20} + 1}{\sqrt{10}}$

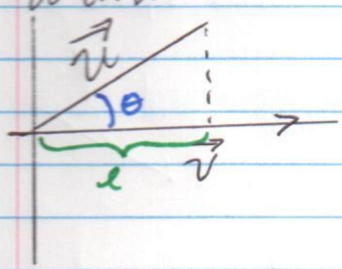
$$\cos\theta = \frac{\frac{-2\sqrt{20}+1}{\sqrt{10}}}{\frac{\sqrt{3}}{1} \cdot \frac{\sqrt{41}}{10}} = \frac{-2\sqrt{20}+1}{\sqrt{10}} \cdot \frac{10}{\sqrt{123}}$$

$$= \frac{-20\sqrt{20} + 10}{\sqrt{1230}}$$

Dot product properties:

1) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$
2) if $\vec{u}\perp\vec{v}$ $(\cos\theta=0)$ $\vec{u}\cdot\vec{v}=0$
3) $\vec{u}\cdot\vec{v}\cdot\vec{w}$ doesn't have any meaning
4) $\vec{u}\cdot\vec{u} = a^2+b^2 = |\vec{u}|^2$
5) $\vec{u}\cdot\vec{0}=0$
6) $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$

The main application of the dot product is finding the projection of $\vec{u}$ on $\vec{v}$.




$\cos\theta = \dfrac{l}{|\vec{u}|}$ $\quad$ $l = |\vec{u}|\cos\theta$ $\quad$ $l = |\vec{u}|\cdot\boxed{\dfrac{\vec{u}\cdot\vec{v}}{\cancel{|\vec{u}|}\cdot|\vec{v}|}} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$

$\cos\theta = \boxed{\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}}$ $\quad$ $l = \text{proj}_{\vec{v}}\,\vec{u}$