Graphs of fractional functions

Eg: $\frac{1}{x}$ Dom $x \in \mathbb{R}, x \neq \emptyset$ "Sketch the graph of the function":
$\mathbb{R} \setminus \{0\}$

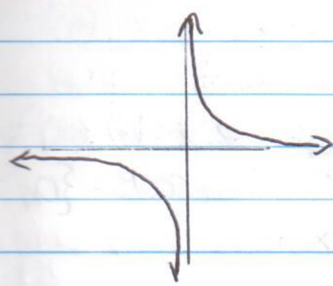
- extremes
- inflection pts
- concavity
- boundary pts in the table.

$f'(x) = -\frac{1}{x^2}$ $f''(x) = \frac{(0 \cdot x^2) - (-1 \cdot 2x)}{x^4} = \frac{2x}{x^4} = \frac{2}{x^3}$

x	$-\infty$	0	$+\infty$
$f'(x)$	$-$	‖	$-$
$f(x)$	↘	‖	↘
$f''(x)$	$-$	‖	$+$
$f(x)$	⌒	‖	⌣

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$ $\lim_{x \to +\infty} \frac{1}{x} = 0$



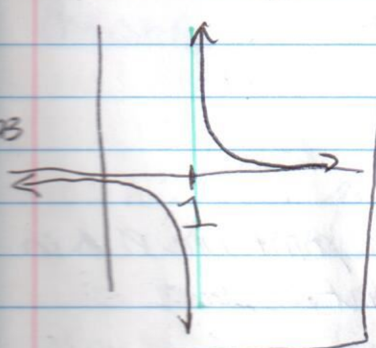
every fractional f(x) has a vertical asymptote at its root. There is an inflection point at 0 but f(0) ∄ in the domain of the f(x) f' and f'' ∄ at the inflection point.

Eg $\frac{x}{(x-1)}$ Dom $\mathbb{R}/\{1\}$ $f'(x) = \frac{(1 \cdot (x-1)) - (1 \cdot x)}{(x-1)^2} = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$

x	$-\infty$	1	$+\infty$
$f'(x)$	$-$	‖	$-$
$f(x)$	↘	‖	↘
$f''(x)$	$-$	‖	$+$
$f(x)$	⌒	‖	⌣

$f'' = \frac{-(2(x-1) \cdot -1)}{(x-1)^4}$ $2(x-1) \cdot 1$

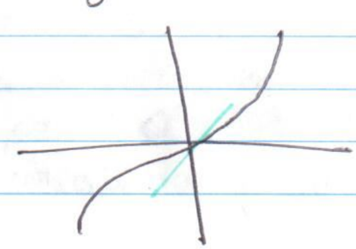
$= \frac{-(-2x+2)}{(x-1)^4} = \frac{2x-2}{(x-1)^4} = \frac{2\cancel{(x-1)}}{(x-1)^4} = \frac{2}{(x-1)^3}$



Eg: x^3 Dom $= \mathbb{R}$ $f'(x) = 3x^2$ $f''(x) = 6x$
$3x^2 = 0$ when $x = 0 \therefore 0 =$ critical point
$3x^2 \geq 0$

x	$-\infty$	0	$+\infty$
$f'(x)$	$+$	0	$+$
$f(x)$	↗	0	↗
$f''(x)$	$-$	0	$+$
$f(x)$	⌒	0	⌣

concavity Δ @ $x=0 \therefore$ it is an inflection point. The tangent line passes through the curve @ this point



Eg: $f(x)\ \dfrac{x^2}{x-2}$ $\quad f'(x) = \dfrac{(2x\cdot(x-2) - (x^2))}{(x-2)^2} = \dfrac{2x^2-4x-x^2}{(x-2)^2} = \dfrac{x^2-4x}{(x-2)^2}$ Dom $x \in \mathbb{R} \setminus \{2\}$

x	$-\infty$		0		2		4		$+\infty$
$f'(x)$		+	0	−	‖	−	0	+	
$f(x)$	$-\infty$	↗	0	↘	$-\infty$ ‖ $+\infty$	↘	8	↗	$+\infty$

$x^2-4x=0 \quad x=4,\ x=0$

$x(x-4)$

$\lim\limits_{x\to-\infty} \dfrac{x^2}{x-2} = -\infty \qquad \lim\limits_{x\to+\infty} \dfrac{x^2}{x-2} = +\infty$

$\lim\limits_{x\to2^-} \dfrac{x^2}{x-2} = -\infty \qquad \lim\limits_{x\to2^+} \dfrac{x^2}{x-2} = +\infty$

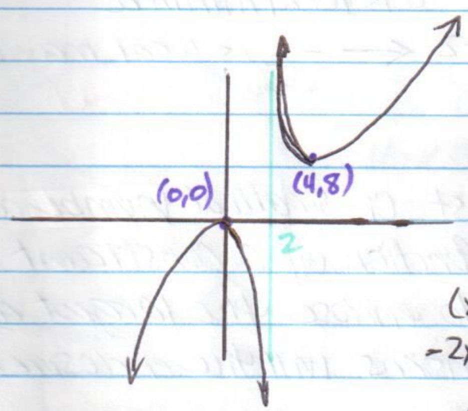
$\dfrac{0^2}{0-2} = 0 \qquad \dfrac{4^2}{4-2} = \dfrac{16}{2} = 8$

$f''(x) = \dfrac{((2x-4)\cdot(x-2)^{\cancel{2}\,1}) - (2\cancel{(x-2)}(x^2-4x))}{(x-2)^{\cancel{4}\,3}} = \dfrac{((2x-4)(x-2)) - (2(x^2-4x))}{(x-2)^3}$ $\quad 2x^2-8x$

$= \dfrac{\cancel{2x^2} - \cancel{4x} - \cancel{4x} + 8 - \cancel{2x^2} + \cancel{8x}}{(x-2)^3} = \dfrac{8}{(x-2)^3}$

x	$-\infty$		2		$+\infty$
$f''(x)$		−	‖	+	
$f(x)$		∩	‖	∪	

2 is not an inflection point because $f(2)\ \nexists$ but 2 is a vertical asymptote.



Eg: $\dfrac{x^2}{(x-1)^2}$ $\quad$ Dom $= \mathbb{R} \setminus \{1\}$

$f'(x) = \dfrac{(2x(x-1)^{\cancel{2}\,1}) - (2\cancel{(x-1)}\cdot x^2)}{(x-1)^{\cancel{4}\,3}} = \dfrac{\cancel{2x^2} - 2x - \cancel{2x^2}}{(x-1)^3} = \dfrac{-2x}{(x-1)^3}$

x	$-\infty$		0		1		$+\infty$
$-2x$		+		−	‖	−	
$(x-1)^3$		−		−	‖	+	
$-2x/(x-1)^3$		−		+	‖	−	
$f(x)$	1^+	↘	0 local min	↗	$+\infty$ ‖ $+\infty$	↘	1^+

$\lim\limits_{x\to-\infty} \dfrac{x^2}{(x-1)^2} \approx \dfrac{x^2}{x^2} = 1^+$

$\lim\limits_{x\to+\infty} \dfrac{x^2}{(x-1)^2} \approx \dfrac{x^2}{x^2} = 1^+$

$\lim\limits_{x\to1^-} \dfrac{x^2}{(x-1)^2} = +\infty \qquad \lim\limits_{x\to1^+} \dfrac{x^2}{(x-1)^2} = +\infty$

$\dfrac{0^2}{0-1^2} = 0$

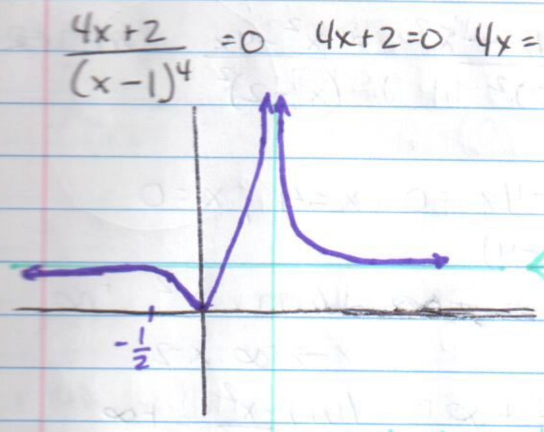
$f''(x) = \dfrac{(-2\cdot(x-1)^{\cancel{3}\,1}) - (3\cancel{(x-1)^2}\cdot -2x)}{(x-1)^{\cancel{6}\,4}}$

$= \dfrac{-2(x-1) - (-6x)}{(x-1)^4} = \dfrac{-2x+2+6x}{(x-1)^4} = \dfrac{4x+2}{(x-1)^4}$

$\frac{4x+2}{(x-1)^4} = 0 \quad 4x+2=0 \quad 4x=-2 \quad x=-\frac{2}{4} = -\frac{1}{2}$

x	$-\infty$	$-1/2$		1	
$f''(x)$	$-$	0	$+$	‖	$+$
$f(x)$	∩		∪	‖	∪

inflection pt



← horizontal asymptote @ $y=1$

vertical asymptote @ $x=1$

Second derivative gives us local extreme points

Second derivative test for local extreme points

If you aren't sketching the graph then this test will give you the local extreme points but it only works for critical points where $f'(x)=0$
If $f'(x) \nexists$ @ that critical point, this test doesn't work @ that point
If $f'(c)$ is 0, $(c, f(c))$ is a critical point. If $f''(c)$ is ⊕ @ this point then $(c, f(c))$ is a local minimum. If $f''(c)$ is ⊖ then $(c, f(c))$ is a local max.

Eg: $f(x) = x^3 + x^2 + 1$
$f'(x) = 3x^2 + 2x \Rightarrow x(3x+2) \quad x=0$ critical points: $\begin{cases} 0 \\ -\frac{2}{3} \end{cases}$
$3x+2=0 \quad 3x=-2 \quad x=-\frac{2}{3}$

$f''(x) = 6x+2 \quad \begin{cases} 6 \cdot 0 + 2 = 0 + 2 = 0 \leftarrow 0 \text{ is local minima} \\ 6 \cdot -\frac{2}{3} + 2 = -\frac{12}{3} + 2 = -4 + 2 = -2 \leftarrow -\frac{2}{3} \text{ is local maxima} \end{cases}$

Optimization:

Optimization is about figuring out the largest or smallest value that a function can take. Sometimes this is a function w/ a constraint.
Eg: a farmer has 800m of fence & wants to enclose the largest area possible in a rectangular shape. What dimensions will the enclosure have?
A = length x width
perimeter = 2(length) + 2(width) = 800
length = $\frac{800 - 2(\text{width})}{2}$ = 400 − width.

A = (400 − width) x width = 400·width − width^2 ($400w - w^2$)

$A(w) = 400w - w^2$ $A'w = 400 - 2w$ ← take the first derivative

find the critical point of the f(x): $400 - 2w = 0$ $400 = 2w$ $\frac{400}{2} = w = 200$

$\therefore A(w)\ max = 200 \times 200 = 40{,}000\ m^2$

L14
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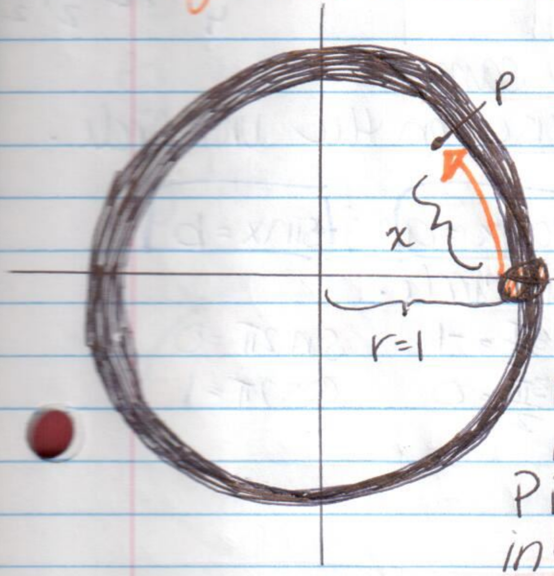
Introduction to trigonometric functions

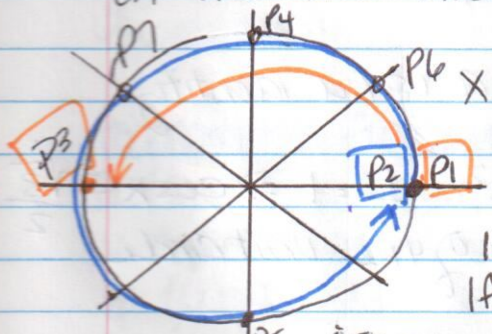
trigonometric functions = $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$



$x^2 + y^2 = r^2$ = equation of a circle with radius r

unit circle = a circle with radius = **1**

imagine that the unit circle centers on the cartesian plane with (0,0) at the center. Suppose we are trying to find an arc on the circumference of the circle where the length of the arc = x. Starting from ⊗ we move counter clockwise across the length = x. P is the terminal point. The point P has coordinates in the cartesian plane. The maximum length of an arc drawn this way is the circumference of the circle. The equation for the circumference of a circle is $2\pi r$. If r=1 then the circumference of the circle is 2π. This makes $x^2+y^2=1$. This is true for any point on the unit circle.



x is both the length of the arc and = the angle

If we start @ $\boxed{P1}$ and don't move, $\boxed{P_1}$ has the coordinates (1,0)

$x=0 \therefore p=(1,0)$ $(1^2+0^2=1)$

If we make a full revolution, $\boxed{P_2} = (1,0)$, $x = 2\pi$

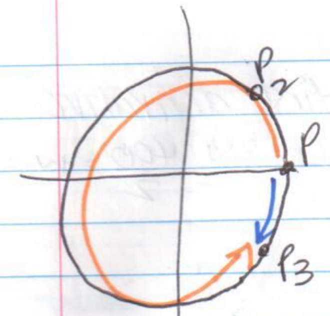
If we make a half a revolution, $\boxed{P_3} = (-1,0)$, $x=\pi$

& so on: ¼ $(P_4 = 0,1, x = \frac{\pi}{2})$ ¾ $(P_5 = 0,-1)$ $x = \frac{3\pi}{2}$

$\frac{1}{8}$ revolution: $x = \frac{\pi}{4}$ $p_6 = (x,y)$? $x=y$ $x^2+b^2=1^2$ $\Rightarrow a^2+a^2=1$

$2a^2 = 1 \Rightarrow a^2 = \frac{1}{2}$ $a = \pm\frac{\sqrt{1}}{\sqrt{2}}$ (rationalize) $\pm\frac{\sqrt{2}}{2}$

P6 is ⊕x, ⊕y $\therefore \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ P7 has ⊖x & ⊕y $\therefore \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ & so on



P is the terminal point for all even multiples of π.
2π, 4π, 6π, or as a formula, 2kπ where k∈N. 2kπ means to travel counterclockwise on the unit circle "k" time
For angles like $\frac{9\pi}{4}$, remove multiples of 2π & examine the

$\frac{3\pi}{2}$ $\frac{-\pi}{2}$, $\frac{9\pi}{4} - \frac{8\pi}{4} = \frac{\pi}{4}$ $\frac{8\pi}{4} = 2\pi$ so 1 full revolution + $\frac{\pi}{4}$. $P_2 = \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$

the terminal points for x+2π & x are the same.
Negative numbers just mean we move clockwise on the unit circle.

Definition: $\boxed{p=(a,b)}$ related to the angle x; $\boxed{\cos x = a}$ & $\boxed{\sin x = b}$
they relate to the terminal point of the unit circle.

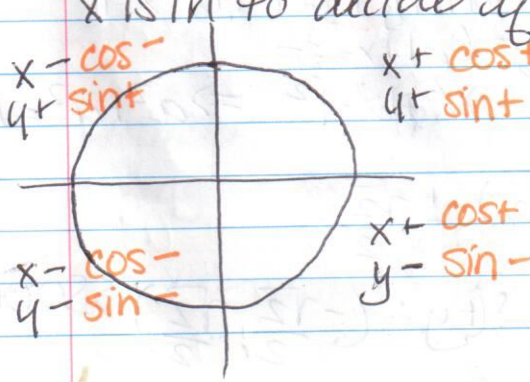
$\sin 0 = 0$ $\sin\frac{\pi}{2} = 1$ $\sin\pi = 0$ $\sin\frac{3\pi}{2} = -1$ $\sin 2\pi = 0$
$\cos 0 = 1$ $\cos\frac{\pi}{2} = 0$ $\cos\pi = -1$ $\cos\frac{3\pi}{2} = 0$ $\cos 2\pi = 1$
$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ & so on etc.

Definition: tangent(x) $\frac{\sin x}{\cos x}$ secant x $= \frac{1}{\cos x}$

cotangent(x) $\frac{\cos x}{\sin x}$ cosecant $\frac{1}{\sin x}$

The equation for a circle is $a^2+b^2=1$
$a=\cos$ & $b=\sin$ ⇒ $\boxed{\cos^2 x + \sin^2 x = 1}$ ← a useful identity

Eg: $\sin x = \frac{1}{2}$. ? cos x ? ⇒ $\cos^2 x + \left(\frac{1}{2}\right)^2 = 1$ ⇒ $\cos^2 x + \frac{1}{4} = 1$ ⇒ $\cos x = \pm\frac{\sqrt{3}}{2}$
then we need more info about what part of the unit circle x is in to decide if it's + or −.



Another useful identity = $\boxed{1+\tan^2 x = \sec^2 x}$

proof:

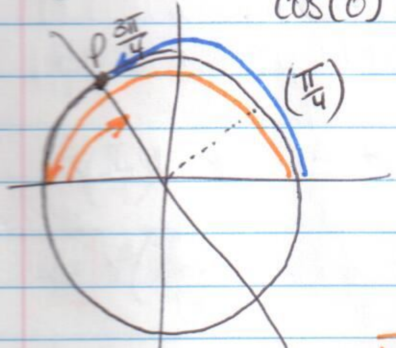
$$\sec^2 x = \frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \boxed{\tan^2 x + 1 = \sec^2 x}$$

$$\boxed{\csc^2 x = 1 + \cot^2 x} = \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} = \csc^2 x$$

Eg: $\tan^2 \frac{\pi}{4} = \sec^2 \frac{\pi}{4} - 1 = \left(\frac{2}{\sqrt{2}}\right)^2 - 1 = \frac{4}{2} - 1 = \frac{4}{2} - \frac{2}{2} = \frac{2}{2} = 1.$

Eg: $\sin \frac{1}{2} = \cos^2 x + \frac{1}{4} = 1 \Rightarrow \cos^2 x = \frac{3}{4} \Rightarrow \cos x = \frac{\sqrt{3}}{2} \pm$

Eg: $\tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$ $\quad \text{cotan}(0) = \frac{\cos(0)}{\sin(0)} = \frac{1}{0} =$ undefined



$\sin 2\pi + x = \sin x$

$\cos 2\pi + x = \cos x$

$\tan 2\pi + x = \tan x$

$\cot 2\pi + x = \cot x$

& so on

Eg: $\boxed{x = \frac{3\pi}{4}}$ or $\boxed{\pi - \frac{\pi}{4}}$

$\frac{3\pi}{4} = (\boxed{-a}, b)$ $\quad \frac{\pi}{4} = (\boxed{a}, b)$

so $\boxed{a =} \cos \frac{\pi}{4}$ & $\boxed{-a} = \cos \frac{3\pi}{4}$

therefore $\cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{4}\right)$ & $\sin\left(\pi - \frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{4}\right)$

this means that $\cos \pi \pm x = \pm \cos x$ $\quad \sin -x = -\sin x$

$\sin \pm x = \pm \sin x$ $\quad \cos -x = \cos x$

trigonometric functions

For any $\mathbb{R}$ number we can find a point on the unit circle.
y depends on x which makes this a function.

$y = f(x) = \sin(x)$ $\quad y = f(x) = \tan(x)$ $\quad y = f(x) = \sec(x)$

$y = f(x) = \cos(x)$ $\quad y = f(x) = \cot(x)$ $\quad y = f(x) = \csc(x)$

remember that some of these functions are related as quotients & this will affect their domains.

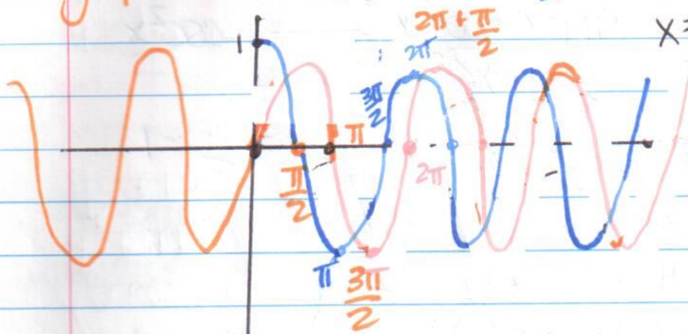
Dom $\sin x = \mathbb{R}$ $\quad$ Dom $\tan x = \frac{\sin x}{\cos x} = x \in \mathbb{R}\ \cos x \neq 0$

Dom $\cos x = \mathbb{R}$

$\cos \frac{\pi}{2} + \frac{2k\pi}{2} = 0$ so Dom $\tan x = \mathbb{R} \setminus \left\{ k\pi + \frac{\pi}{2} \right\}$ $k \in \mathbb{Z}$

Dom $\cot(x)$ = dom $\frac{\cos(x)}{\sin(x)}$ $\sin \neq 0$ Dom $\mathbb{R} \setminus \{k\pi\}$ $k \in \mathbb{Z}$

graph of $\sin(x)$ $\cos(x)$



$x=0$, $\sin x = 0$ $(0,0)$
$x=\frac{\pi}{2}$ $\sin x = 1$ $(\frac{\pi}{2}, 1)$
$x=\pi$ $\sin x = 0$ $(\pi, 0)$
$x=\frac{3\pi}{2}$ $\sin x = -1$ $(\frac{3\pi}{2}, -1)$
$x=2\pi$, $\sin x = 0$ $(2\pi, 0)$

these are periodic functions with a period = 2π.

Derivatives of trigonometric functions:

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some preliminary derivatives:

$$\boxed{\begin{array}{ll} f(x) = \sin x & f'(x) = \cos x \\ f(x) = \cos x & f'(x) = -\sin x \end{array}}$$

Eg: find the derivative of $\tan x$

$f(x) = \tan x = \frac{\sin x}{\cos x}$ $\quad f'(x) = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{\cos^2 x}$

pythagorean identity

$f'x = \frac{(\cos x \cdot \cos x) - (\sin x \cdot -\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$

? $f'(x)$ of $f(x) = \cot x$?

$f(x) = \cot x = \frac{\cos x}{\sin x}$ $\quad f'(x) = \frac{(\cos x)'(\sin x) - (\cos x)(\sin x)'}{\sin^2 x}$

$f'(x) = \frac{(-\sin x \cdot \sin x) - (\cos x \cdot \cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$f'x = \frac{-1}{\sin^2 x} = -\left(\frac{1}{\sin x}\right)^2 = -\csc^2 x = -(1 + \cot^2 x)$

$f(x) = \boxed{\sec x}$ $f'(x) = ?$ $\quad 0 \cdot \cos x \quad -\sin x \cdot 1$

$f(x) = \sec x = \frac{1}{\cos x}$ $\quad f'(x) = \frac{(1' \cdot \cos x) - (\cos x' \cdot 1)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \circ \frac{1}{\cos x} = \sec x \tan x$

$f(x) = \boxed{\csc x}$ $f'(x) = ?$

$f(x) = \csc x = \frac{1}{\sin x}$ $\quad f'(x) = \frac{(0 \cdot \sin x) - (\cos x \cdot 1)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\left(\frac{\cos x}{\sin x}\right) \circ \frac{1}{\sin x}$

$= -\cot x \circ \csc x$

Eg: what is the equation for the line tangent to the curve $y = \sin x \cos x$ at the point $\frac{\pi}{4}$?

$y = mx + b$ $\quad \boxed{m} = f'\left(\frac{\pi}{4}\right)$ $\quad f'(x) = (\cos x \cdot \cos x) + (-\sin x \cdot \sin x)$

$= \cos^2 x - \sin^2 x$

$f'\left(\frac{\pi}{4}\right) = \cos^2 \frac{\pi}{4} - \sin^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 0 = m$

$\boxed{y =} f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} \circ \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \circ \frac{\sqrt{2}}{2} = \frac{2}{4} \boxed{= \frac{1}{2}}$

$\frac{1}{2} = 0 \cdot \frac{\pi}{4} + b$ $\quad \boxed{\frac{1}{2} = \boxed{b}}$ $\quad \boxed{y = \frac{1}{2}}$ equation for the line

find the derivative:

$f(x) = \sin^{100} x$ $\quad f(x) = (u_x)^n$ $\quad f'(x) = n u' u^{n-1}$

$n = 100$ $\quad u = \sin x$

$n-1 = 99$ $\quad u' = \cos x$ $\quad f'(x) = 100 \cos x \sin^{99} x$

$\boxed{\text{in general } y = \sin^n x \quad y' = n \cos x \sin^{n-1} x}$

$f(x) = \cos^3 x$ $\quad f'(x) = 3(-\sin x) \cos^2 x$

$\boxed{\text{in general } y = \cos^n x \quad y' = n(-\sin x)(\cos x)^{n-1}}$

$\boxed{f(x) = \sec^n x \quad y' = n \sec x \tan x \sec^{n-1} = n \sec^n x \tan x}$

$\boxed{y = \csc^n x \quad y' = -n\csc x \cot x \csc^{n-1} x = -n\csc^n x \cot x}$

$y = \sqrt{\tan x} = (\tan x)^{\frac{1}{2}}$
$y' = \frac{1}{2}\sec^2 x \tan^{-\frac{1}{2}} = \frac{\sec^2 x}{2\sqrt{\tan x}}$

$y = \sin 3x \quad y' = f'(gx)\circ g'x$
$y' = (\sin'(3x))\circ(3x')$
$y' = \cos 3x \circ 3 = 3\cos 3x$

$\boxed{\text{in general } y = \sin(u) \quad y' = u'\cos u}$
$\boxed{y = \cos(u) \quad y' = u' - \sin u}$

$y = \sin(x^2 + 2x)$
$y' = (2x+2)(\cos(x^2+2x))$

$y = \cos(x^2+2x)$
$y' = (2x+2)^\circ - \sin(x^2+2x)$

$\boxed{y' \tan u = u'\sec^2 u}$ $\boxed{y' \cot u = -u'\csc^2 u}$ $\boxed{y' \sec u = u'\sec u \tan u}$

Eg: $y = \sec(\sqrt{x}) \qquad y' = \frac{1}{2\sqrt{x}}\sec(\sqrt{x})\tan(\sqrt{x})$

$y = \csc(\sin x) \qquad y' = \cos x(-\csc(\sin x)\cot(\sin x))$
$= -\cos x \csc(\sin x)\cot(\sin x)$

$y = \frac{\sec(x^2+\sqrt{x})}{\tan^3 x} \qquad y' = \frac{((\sec(x^2+\sqrt{x}))'\circ(\tan^3 x)) - ((\tan^3 x)'\circ\sec(x^2+\sqrt{x}))}{(\tan^3 x)^2}$

$y' = \frac{2x + \frac{1}{2\sqrt{x}}\sec(x^2+\sqrt{x})\tan(x^2+\sqrt{x})(\tan^3 x) - (3\sec^2 x \circ \sec(x^2+\sqrt{x}))}{\tan^6 x}$

$y = \csc^3(x^2) \qquad y' = nu'u^{n-1}$
$y' = u^3 = 3u'u^{3-1} = 3u'u^2$
$u = x^2 \quad u' = 2x$
$u = (x^2)'(-\csc x^2 \cot x^2) = 2x - \csc x^2 \cot x^2$
$y' = 3(2x - \csc x^2 \cot x^2 \csc^2 x^2) = 6x\csc^3 x^2 \cot x^2$

chain rule x2

$y = \cot \frac{1}{x}$ $u = \frac{1}{x}$ $u' = -x^{-2}$

$(\cot x)' = \csc^2 x$ $y' = \csc^2 \frac{1}{x} \cdot -x^{-2} = \frac{1}{x^2} \cdot \csc^2(\frac{1}{x}) = \frac{\csc^2(\frac{1}{x})}{x^2}$

Exponential functions

Lec16 Nov7/2013 Exponential maps are about the rate of growth of a fast movement
Eg: the rate of growth of a bacterial culture
: astronomical motions

The variable is the exponent

Eg: the population growth of a colony of rabbits is expressed as $i \cdot 2^x$ where i is the initial number and x is the number of years.
If $i = 5$ the function is $N(t) = 2^t$ (t = years)

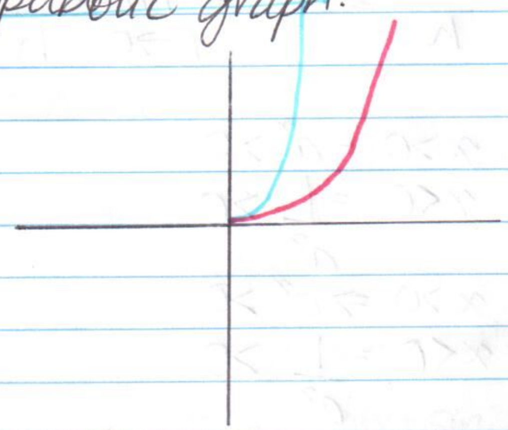
Initial: $5 \cdot 2^0 = 5$
1 year: $5 \cdot 2^1 = 10$
2 years: $5 \cdot 2^2 = 20$
3 years: $5 \cdot 2^3 = 40$
4 years: $5 \cdot 2^4 = 80$
n years $= 5 \cdot 2^n$

in general an exponential is a function with the form $f(x) = a^x$ where a is a constant

contrast this with polynomials where the base is the variable & the exponent is fixed:

1	$1^2=1$	$2^1=2$
2	$2^2=4$	$2^2=4$
3	$3^2=9$	$2^3=8$
4	$4^2=16$	$2^4=16$
5	$5^2=25$	$2^5=32$
6	$6^2=36$	$2^6=64$
7	$7^2=49$	$2^7=128$
8	$8^2=64$	$2^8=256$
9	$9^2=81$	$2^9=512$
10	$10^2=100$	$2^{10}=1024$

the graph of the exponential function is called a hyperbolic graph:



a is alway considered to be positive.

$y = a^x \begin{Bmatrix} a>1 \\ 0<a<1 \end{Bmatrix}$ 2 cases:

1) $a>1$ $a^x \nearrow$ as $x \to \infty$

$\lim_{(x\to\infty)} a^x = \infty$ $\lim_{\substack{(x\to -\infty) \\ \text{or} \\ (-x\to\infty)}} a^{-x} = \frac{1}{a^x} = \frac{1}{+\infty} = 0^+$

2) $0<a<1$
if $a = \frac{1}{2}$ $\longrightarrow$ $a^1 = \frac{1}{2}$ $a^2 = \frac{1}{4}$ $a^3 = \frac{1}{8}$ $a^4 = \frac{1}{16}$ $\therefore \lim_{x\to\infty} a^x = 0^+$

Eg: $\left(\frac{1}{2}\right)^{-10} = 2^{10}$ $a = \left(\frac{1}{5}\right)^{-1} = \frac{1}{(1/5)} = 1 \cdot \frac{5}{1} = 5$ $\lim_{\substack{x\to -\infty \\ (-x\to\infty)}} a^{-x} = \frac{1}{a^{+\infty}} = \frac{1}{0^+} = +\infty$

$\left(\frac{1}{5}\right)^{-2} = \frac{1}{(1/5)^2} = \frac{1}{1/25} = 1 \cdot \frac{25}{1} = 25$

$y = e^x$ e is similar to π in that it is an irrational $\mathbb{R}$ number.

$e = \left(1 + \frac{1}{n}\right)^n_{n\to\infty}$ $\boxed{y'(e^x) = e^x !}$

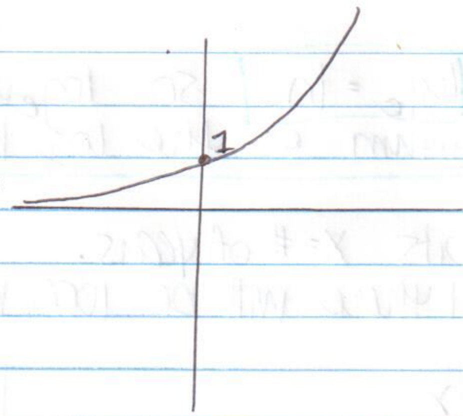
proof: $\lim_{h\to 0} \frac{e^{(x+h)} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} = \frac{1 \cdot e^x}{1} = e^x$

$a>1$ $a^x > 0$ $x>0 = a^x > 0$
$x<0 = \frac{1}{a^x} > 0$

$0<a<1$ a^x: $x>0 \Rightarrow a^x > 0$
$x<0 = \frac{1}{a^x} > 0$

$y=e^x \quad y'=e^x \quad y''=e^x$

x	$-\infty$	0	$+\infty$
$f'(x)$	$+$		$+$
$f(x)$	$\nearrow$	1	$\nearrow$
$f''(x)$	$+$		$+$
$f(x)$	$\cup$		$\cup$



$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$ Eg: $y=e^{x^2}$ $y'=2xe^{x^2}$

in general, $\boxed{y=e^u}$ $\boxed{y'=u'e^u}$

$f(x)=e^x \quad u'=e^x$

$g(x)=u \quad u'=u$

$y'(x)=(f\circ g)'(x) = f'(g(x))\cdot g'(x)$

$= u'e^u = f'(u)(u')$

eg: $e^{(x^3+\sqrt{x})}$ $y'=\left(3x^2+\frac{1}{2\sqrt{x}}\right)e^{(x^3+\sqrt{x})}$

eg: $e^{\sin x}$ $y'=\cos x\, e^{\sin x}$

eg: $e^{\frac{1}{x}}$ $y'=u'e^u$ $u'=\frac{1}{x^2}$ $y'=\frac{1}{x^2}e^{\frac{1}{x}}$

eg: $y'=e^{(\sin x-\cos^2 x)}$

$u=\sin x-\cos^2 x$

$u'=(\sin x)'-(\cos^2 x)'$

$(\sin x)'=\cos x$

$(\cos^2 x)'=nu'u^{n-1}$

$=n=2 \quad u'=-\sin x \quad u=\cos x \quad n-1=1$

$(\cos^2 x)'=2 -\sin x\cos x$

$=-2\sin x\cos x$

$u'=\cos x+2\sin x\cos x$

$y'=(\cos x+2\sin x\cos x)e^{(\sin x-\cos^2 x)}$

remember that $\boxed{\sin 2x=2\sin x\cos x}$ $\rightarrow y'=\cos x+\sin 2x\, e^{(\sin x-\cos^2 x)}$

Eg: $y=e^{\sqrt{\tan x}}$ $u=\sqrt{\tan x}$ $u'=\frac{1}{2\sqrt{\tan x}}\cdot\sec^2 x=\frac{\sec^2 x}{2\sqrt{\tan x}}$

$y'=\frac{\sec^2 x}{2\sqrt{\tan x}}e^{\sqrt{\tan x}}$