Notation

$$f'(x) = \lim_{h\to 0} \frac{f(a+h) - a}{h} = \lim_{\substack{\Delta a\to 0\\ h\to 0}} \frac{\Delta f}{\Delta a} \quad ((a+h)-a) = h \quad a \to 0$$

Sometimes instead of writing $\lim_{h\to 0} \frac{\Delta f}{\Delta a}$ we just write $\frac{df}{da}$ (or $\frac{dy}{dx}$ or $\frac{df}{dx}$ ...)

when we use this notation $\lim_{h\to 0}$ is implied.

important!

The chain rule

$f\circ g'(x)$ or $f'(g(x)) = f'(g(x)) \cdot g'(x)$

leibniz notation $= \frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Eg $= (x^2+x)^{100}$

$f(x) = x^{100} \quad f'(x) = 100x^{99}$

$g(x) = x^2+x \quad g'(x) = 2x+1$

$\therefore f'(g(x)) \circ g'(x) = 100(x^2+x)^{99} \circ 2x+1$

Eg: $\sqrt{x^2+x} = (x^2+x)^{\frac{1}{2}}$

$f(x) = x^{\frac{1}{2}} \quad f'x = \frac{1}{2\sqrt{x}}$

$g(x) = x^2+x$

$g'(x) = 2x+1$

$f\circ g'(x) \quad \frac{1}{2\sqrt{x^2+x}} \circ \frac{2x+1}{1} = \frac{2x+1}{2\sqrt{x^2+x}}$

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Sometimes instead of $g(x)$ in $f'(g(x)) \cdot g'(x)$ we just call it "u"

$(f(u))' = f\circ g(x) = f'(u) \circ u'$

or

So for something like $(2x+1)^{\frac{1}{7}}$ we call it $u^{\frac{1}{7}}$

where $h(x) = (u)^n \quad f'(x) = nu^{n-1}$

$f(x) = x^n \quad g'(x) = u'$

$g(x) = u \quad \boxed{h'(x) = nu'u^{n-1}}$ ☆important!☆

lec 10 Physics example for the application of derivatives

$h =$ $s(t) = 90 - 4.9t^2$

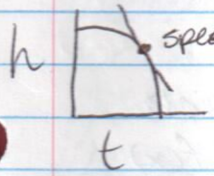
determine the average velocity of the falling object between [1 4] sec

velocity = $\frac{\text{distance}}{\text{time}}$ or average R.O.C. between 1 & 4 seconds

$$\frac{\Delta h}{\Delta t} = \frac{(90 - 4.9 \cdot 4^2) - (90 - 4.9 \cdot 1^2)}{4-1} = \frac{(90 - 4.9 \cdot 16) - (90 - 4.9)}{3} = -24.5$$

$-$ because the vector is $\downarrow$

what is the velocity @ $t = 4$ (instantaneous ROC @ $t=4$)



speed = slope of tangent line or the derivative of $s(t)$

$s't = -9.8t$ $\quad s'(4) = -9.8 \cdot 4 = \underline{-39.2} = v(t)$

what is the acceleration @ $t=3$?

= instantaneous ROC of velocity or $v'(t)$

if $v(t)$ is $s'(t)$ then $v'(t)$ would be $s''(t) = (s'(t))'$ "double prime"

$v'(t) = a(t)$ (acceleration)

$a(t) = s''(t) = -9.8 = v'(t)$ for $v(t) = -9.8t$

Eg: $s(t) - 7t^3 + 5t^2 + 1$

$s'(t) = 21t^2 + 10t = v(t)$

$v't = s''(t) = 2 \cdot 21t + 10 = 42t + 10$

leibniz notation:

$y' = \frac{dy}{dx}$ $\quad y'' = \frac{d^2y}{dx^2}$

we can take a 3rd derivative too. The notation is $y^{(3)} = \frac{d^3y}{dy^3}$

more is $y^{(n)} = \frac{d^ny}{dy^n}$

Chapter 3: the applications of derivatives

increasing and decreasing functions:

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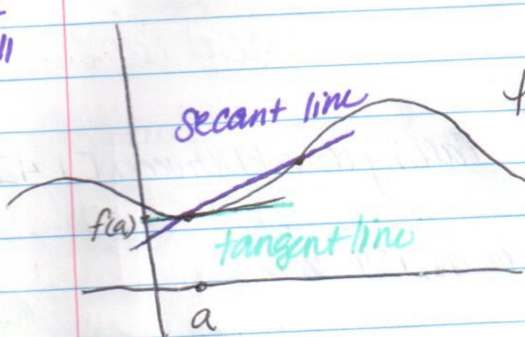
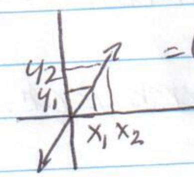
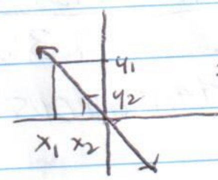


the slope of the tangent line is f'a



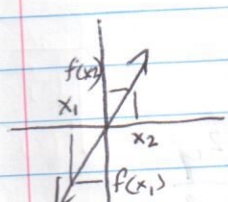
= (+)ve slope
or $m = \dfrac{y_2 - y_1}{x_2 - x_1} > 0$
$f'(x) =$



= (−)ve slope
or $m = \dfrac{y_2 - y_1}{x_2 - x_1} < 0$
$f'(x) =$

We say that the function is <u>increasing</u> over the set of points $(a,b)(x_1, x_2) = I$ if for any two points I have $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
& <u>decreasing</u> if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

We only talk about increasing or decreasing with regards to a <u>segment</u> of the function.



$y = mx + c$ = a linear function
↑ slope ↑ constant

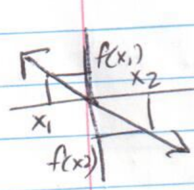
$x_1 < x_2$ & $f(x_1) < f(x_2)$ ∴ fx is <u>increasing</u>.

$f(x) = mx + b$
$f'(x) = m$

We can show this on a table

x	$-\infty$	0	$+\infty$
$f'(x)$	+		+
$f(x)$	↗		↗

if $m > 0$ the fx is ↑ increasing for any $x \in \mathbb{R}$.
& vice versa.



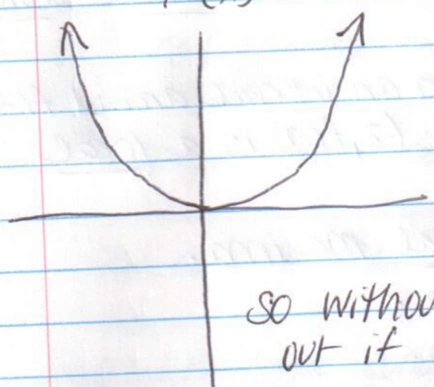
$x_1 < x_2$ & $f(x_1) > f(x_2)$
$y = mx + c$
$f'(x) = m$

if $m < 0$ the fx is ↓ decreasing.

some functions ↑ over some intervals and ↓ over others.

Eg $f(x) = x^2$ $\boxed{\text{Dom } x \in \mathbb{R}}$ let's look at the sign of these functions.

$f'(x) = 2x.$ ← the sign of this fx depends on x



x	$-\infty$	0	$+\infty$ ← the domain of the fx
$f'(x)$	$-$ opposite sign	0	$+$ same sign as the coefficient
	↘		↗

f is ↗ on $(0 \;\; +\infty)$ and ↘ on $(-\infty \;\; 0)$

so without having the graph in front of you you can figure out if $f(x)$ is ↑ing or ↓ing.

Theorem:

for the fx $f(x)$ on the interval $(a \;\; b)$

i) $\forall x \in (a \;\; b)$ $f'(x) > 0 \Rightarrow f(x)$ ↑ on $(a \;\; b)$

ii) $\forall x \in (a \;\; b)$ $f'(x) < 0 \Rightarrow f(x)$ ↓ on $(a \;\; b)$

Eg: $2x^3 + 3x^2 - 36x + 5$ dom $x \in \mathbb{R}$

$f'(x) = 6x^2 + 6x - 36$

$6(x^2 + x - 6)$

$6(x+3)(x-2)$ $\quad x = -3$ for $f'(x) = 0$.
$\quad = +2$

opposite sign of $\underline{6}x^2$

x	$-\infty$	-3		$+2$	$+\infty$
$f'(x)$	$+$		$-$		$+$
$f(x)$	↗		↘		↗

Eg: $(x^2-4)^2 = f(x)$

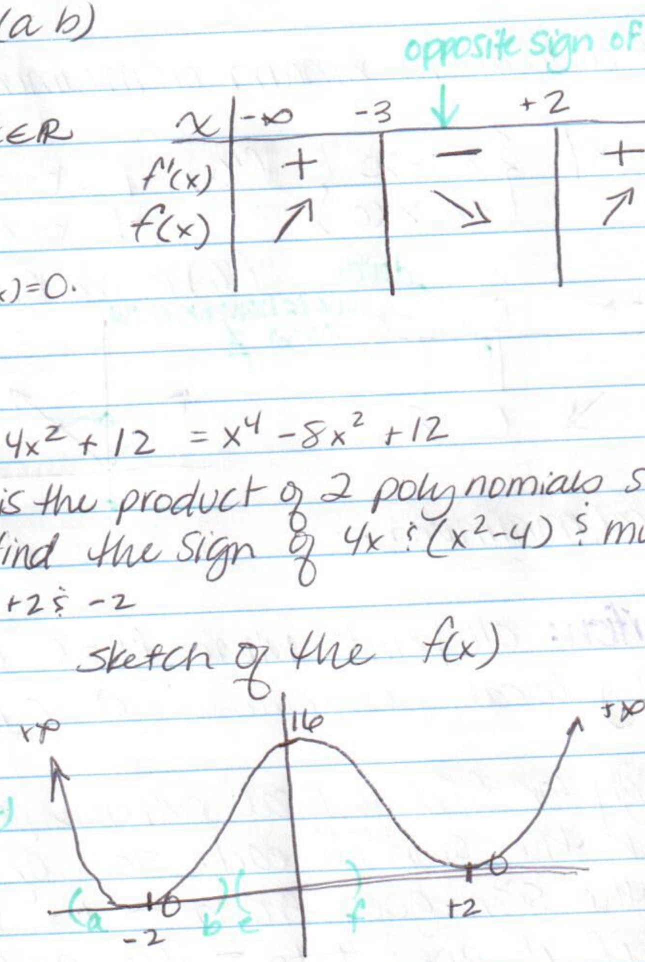
$(x^2-4)(x^2-4) = x^4 - 4x^2 - 4x^2 + 12 = x^4 - 8x^2 + 12$

$f'(x) = 4x^3 - 16x = 4x(x^2-4)$ is the product of 2 polynomials so find the sign of $4x$ & (x^2-4) & multiply together

$(x^2-4) = (x-2)(x+2)$ so roots $= +2$ & -2

x	$-\infty$	-2		0		$+2$	$+\infty$
(x^2-4)	$+$	0	$-$		$-$	0	$+$
$4x$	$-$		$-$	0	$+$		$+$
$4x(x^2-4)$	$-$	0	$+$	0	$-$	0	$+$ ← $f'(x)$
$f(x)$	↘		↗		↘		↗
	$+\infty$	0		16		0	$+\infty$

sketch of the $f(x)$



$f(0) = (-4)^2 = 16$ $\quad f(-2) = (4-4)^2 = 0$
$\quad f(+2)$

$\lim\limits_{x \to +\infty} f(x) = +\infty$ $\quad \lim\limits_{x \to -\infty} f(x) = +\infty$

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$\therefore f(x) \searrow (-\infty \ \ -2) \cup (0 \ \ 2)$
$f(x) \nearrow (-2 \ \ 0) \cup (2 \ \ +\infty)$

$\forall x \in (a \ b) \ f(x) > f(-2)$ • for all values of x on an interval around $f(-2)$ $f(x)$ will always be $> f(-2)$ (>0). this means $(-2, f(-2))$ is a local minima for $f(x)$.
Similarly $\forall x \in (e \ f) \ f(x) < f(0) \therefore f(0)$ is a local maxima.
all local extremes are critical points.

Definition:
$(c, f(c))$ on a graph is a local maxima if we can find @ least one interval around c on the x axis (i) such that $\forall x \in i \ f(x) \leq f(c)$
& a local minima for an interval (i) around c that $\forall x \in i \ f(x) \geq f(c)$

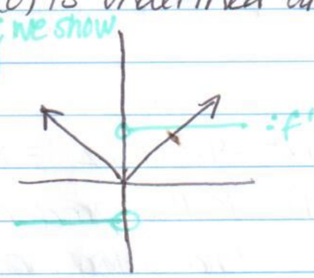
f' of a local maximum or minimum is $= 0$ if it exists.

Eg: $|x| \ \begin{Bmatrix} x & x > 0 \\ -x & x < 0 \end{Bmatrix}$ $\quad f'(x) = 1 \ \ \forall x > 0$
$\qquad\qquad\qquad\qquad\qquad\quad = -1 \ \ \forall x < 0$

$f'(0)$ is undefined and $\nexists$.

double line is how we show $f'(x) \nexists$

x	$-\infty$	0	$+\infty$
$f'(x)$	$-$	$\|\|$	$+$
$f(x)$	$\searrow$	0	$\nearrow$



0 = local minimum.

Definition: all points where $f' = 0$ or $\nexists$ are critical points.
$\therefore$ any local extreme is a critical point.

Strategy for finding local extreme points: 1) find critical points
2) check the sign of f' on each side of the critical point
3) if the sign goes from $-$ to $+$ it's a local minima
& if it goes $+$ to $-$ it's a local maxima

absolute extreme points are the extremes found inside an interval. To find absolute extreme points you have to check all critical points and the edges of the interval.

critical points: $\{x \in Dom\, f : f' = 0 \text{ or } \nexists\}$

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(the set of values for x belonging to the domain of f where $f' = 0$ or doesn't exist)

local extreme points $\subseteq$ critical points
(are contained in the set of)

with an extra condition: the sign of f' on each side of the local extreme is different $\rightarrow$ there is a sign change as it passes through the critical point.

Eg: $\frac{1}{x^2+1}$ Dom $x \in \mathbb{R}$ $f'(x) = \frac{-2x}{(x^2+1)^2}$ Dom $f'(x) = \underline{x \in \mathbb{R}}$ $\therefore$ all critical points $\exists$ & are $=0$

$\frac{-2x}{(x^2+1)^2} = 0$ for $x = 0$. $\therefore x = 0$ is the only critical point.

x	$-\infty$	0	$+\infty$
$f'x$	$+$	0	$-$
$f(x)$	$\nearrow$		$\searrow$

0 = local maxima
no local minima.

What are the absolute extreme points on [2 3]?
0 is not contained in this interval so we don't need to check f(0) only 2 & 3.
(the borders of the interval)

$f(2) = \frac{1}{2^2+1} = \boxed{\frac{1}{5}}$ $f(3) = \frac{1}{3^2+1} = \boxed{\frac{1}{10}}$

absolute maxima on [2 3] absolute minima on [2 3]

on [-1 3] $\leftarrow$ 0 is contained in this set so we also need to ✓ f(0)

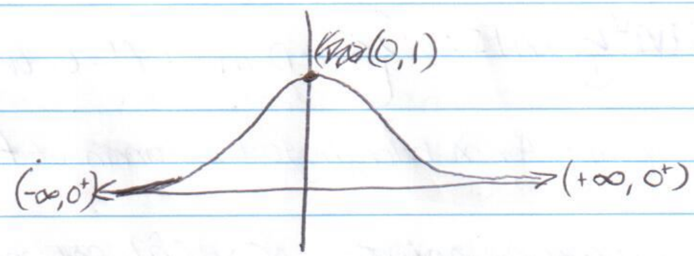
$f(0) = \underbrace{\frac{1}{0^2+1}}_{\text{abs. max. on } [-1\ 3]} = 1$ $f(-1) = \frac{1}{-1^2+1} = \frac{1}{2}$ $f(3) = \frac{1}{3^2+1} = \underbrace{\frac{1}{10}}_{\text{abs. min. on } [-1\ 3]}$

only critical points & ending points can be absolute extremes.

to draw the graph find the limits of X → boundary pts ; f(critical points)

$\lim_{x\to-\infty} \frac{1}{x^2+1} = 0^+$ $\lim_{x\to+\infty} \frac{1}{x^2+1} = 0^+$ $f(0) = \frac{1}{0^2+1} = \frac{1}{1} = 1$

x	$-\infty$	0	$+\infty$
$f'(x)$	+	0	−
$f(x)$	↗		↘
	$+\infty$	1	$+\infty$



Eg: $f(x)\ x - \sqrt{x}$ Dom $x \in \mathbb{R},\ x \geq 0$

$f'(x)\ 1 - \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x} - 1}{2\sqrt{x}}$ $2\sqrt{x} - 1 = 0$ $\sqrt{x} = \frac{1}{2}$ = $x = \frac{1}{4}$

$2\sqrt{x} = 0$ $x = 0$ ∄ = 2 critical points

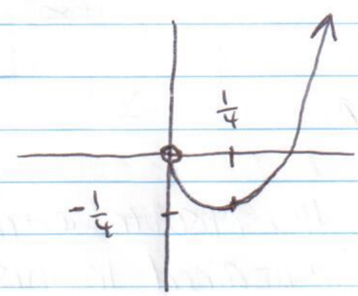
x	0	$\frac{1}{4}$	$+\infty$
f'	−	0	+
f	↘		↗

$2\sqrt{x} - 1 > 0 \Rightarrow \sqrt{x} > \frac{1}{2}$ $x > \frac{1}{4}$ = ⊕

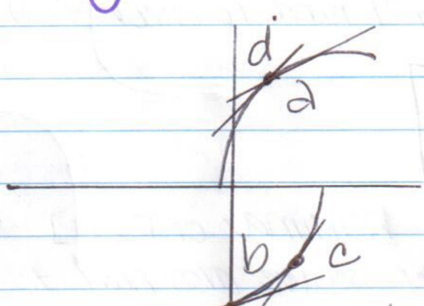
$2\sqrt{x} - 1 < 0 \Rightarrow \sqrt{x} < \frac{1}{2}$ $x < \frac{1}{4}$ = ⊖

$\lim_{x\to0^+} f(x) = 0 - \sqrt{0} = \boxed{0}$

$f\frac{1}{4} = \frac{1}{4} - \sqrt{\frac{1}{4}} = \frac{1}{4} - \frac{1}{2} = \boxed{-\frac{1}{4}}$ $\lim_{x\to\infty} f(x) = \boxed{\infty^+}$



Concavity



- both curve a + b are increasing
- a's tangent line is always above the curve ; b's tangent line is always below the curve
- we only talk about concavity as it relates to a specific point on the graph of the fx

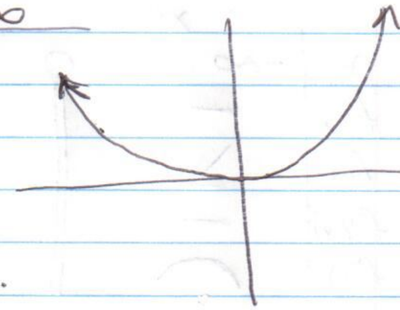
curve b is concave up at point c if the tangent line @(d, f(d)) is below the graph ; a's concave down @ point d if the tangent line @(d, f(d)) is above the graph. This allows us to sketch the graph more accurately.

to check for concavity we check the sign of f''

rule: if $f''c$ is ⊕ then f is concave up @ $(c, f(c))$
if $f''c$ is ⊖ then f is concave down @ $(c, f(c))$

Eg $f = x^2$
$f'x = 2x$
$f''x = 2$ ∴ concave up.

x	$-\infty$	0	$+\infty$
$f'(x)$	$-$		$+$
$f(x)$	↘		↗
$f''(x)$	$+$		$+$
$f(x)$	∪		∪



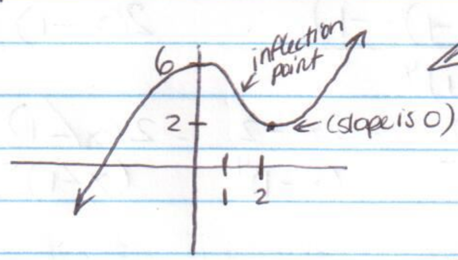
∪ is the notation for concave up. ∩ is the notation for concave down.

Eg: $x^3 - 3x^2 + 6$
$f' = 3x^2 - 6x$ $f' = 0 = 3x^2 - 6x = 0 = 3x^2 = 6x = \frac{6}{3} = \frac{x^2}{x} = 2$
$f'' = 6x - 6$ $= x(3x - 6) = 0$ for $x = 0$ roots $= \{0, 2\}$

$6x - 6 = 0$ $6x = 6$ $x = 1$

x	$-\infty$	0		1		2	$+\infty$
$f'(x)$	$+$	0	$-$		$-$	0	$+$
$f(x)$	↗	0	↘		↘	0	↗
$f''(x)$	$-$		$-$		$+$		$+$
$f(x)$	∩		∩		∪		∪

the point where concavity changes is the inflection point. If f'' exists at the inflection point, it is equal to $\emptyset$.



Eg: x^4 dom $x \in \mathbb{R}$
$f' = 4x^3$
$f'' = 12x^2 \rightarrow$ always ⊕

x	$-\infty$	0	$+\infty$
f'	$-$	0	$+$
f	↘		↗
f''	∪	0	∪

f'' is 0 @ $f''(0)$ but 0 is not an inflation point b/c $\emptyset$ Δ in concavity. This graph has no inflection point.

Graphs of fractional functions

Eg: $\frac{1}{x}$ Dom $x \in \mathbb{R}, x \neq 0$ "sketch the graph of the function"

$\mathbb{R} \setminus \{0\}$

= – extremes
– inflection pts
– concavity
– boundary pts
in the table.

$f'(x) = -\frac{1}{x^2}$ $f''(x) = \frac{(0 \cdot x^2) - (-1 \cdot 2x)}{x^4} = \frac{2x}{x^4} = \frac{2}{x^3}$

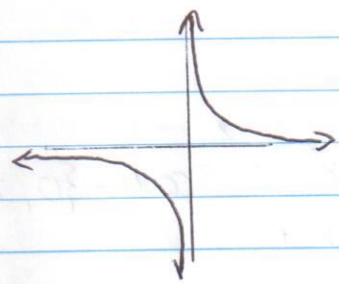
x	$-\infty$	0	$+\infty$
$f'(x)$	$-$	‖	$-$
$f(x)$	↘	‖	↘
$f''(x)$	$-$	‖	$+$
$f(x)$	⌒	‖	◡

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$

$\lim_{x \to \infty} \frac{1}{x} = 0$



every fractional f(x) has a vertical asymptote at its root. There is an inflection point at 0 but $f(0) \nexists$ in the domain of the f(x) f' and f'' $\nexists$ at the inflection point.

Eg $\frac{x}{(x-1)}$ Dom $\mathbb{R} / \{1\}$ $f'(x) = \frac{(1 \cdot (x-1)) - (1 \cdot x)}{(x-1)^2} = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$

x	$-\infty$	1	$+\infty$
$f'(x)$	$-$	‖	$-$
$f(x)$	↘	‖	↘
$f''(x)$	$-$	‖	$+$
$f(x)$	⌒	‖	◡

$f'' = \frac{-(2(x-1) \cdot -1)}{(x-1)^4}$ $2(x-1) \cdot 1$

$= \frac{-(-2x + 2)}{(x-1)^4} = \frac{2x - 2}{(x-1)^4} = \frac{2\cancel{(x-1)}}{\cancel{(x-1)}^4} = \frac{2}{(x-1)^3}$