Mat 1339 Lec 7 Sep 26 Limits & Derivatives

$\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$ if $\frac{0}{0}$ then this is the indeterminate case

$\therefore$ find a factor of $x-a$ in the functions $(f(a)/g(a))$ to cancel it & transform the function to a determinate case @ the point a.

Eg: $\lim\limits_{x\to 8} \frac{2-\sqrt[3]{x}}{8-x} \Rightarrow \lim\limits_{x\to 8} \frac{2-\sqrt[3]{8}}{8-8} = \frac{0}{0}$

we can get rid of the $\sqrt[3]{\ }$ by factoring the fx using diff. of cubes identity

difference of cubes formula: $c^3-d^3=(c-d)(c^2+cd+d^2)$

$\lim\limits_{x\to 8} \frac{2-\sqrt[3]{x}}{8-x} \Longrightarrow \lim\limits_{x\to 8} \frac{(2-\sqrt[3]{x})(4+2\sqrt[3]{x}+\sqrt[3]{x^2})}{(8-x)(4+2\sqrt[3]{x}+\sqrt[3]{x^2})} \to \frac{8-x}{8-x(4+2\sqrt[3]{x}+\sqrt[3]{x^2})}$

$c^3-d^3=(c-d)(c^2+cd+d^2)$ where $2^3=c$ & $(\sqrt[3]{x})^3=d$ and $\frac{2^2+2\sqrt[3]{x}+\sqrt[3]{x^2}}{2^2+2\sqrt[3]{x}+\sqrt[3]{x^2}}=1$

$\lim\limits_{x\to 8} \frac{1}{4+2\sqrt[3]{x}+\sqrt[3]{x^2}} \Rightarrow \lim\limits_{x\to 8} \frac{1}{4+2\cdot\sqrt[3]{8}+\sqrt[3]{8^2}} \Rightarrow \lim\limits_{x\to 8} \frac{1}{4+\underbrace{2\cdot 2}_{4}+4} = \frac{1}{12}$

Eg: $\lim\limits_{x\to 1} \frac{x^2-6x+5}{(x-1)^2} \Rightarrow \frac{0}{0}$ the factor that makes this the indeterminate case is $x-1$: try to find a factor for $x-1$

we can factor by finding 2 numbers where, the sum is -6 and product is $+5$ & we know one is -1 so the 2nd is -5 or we can use polynomial long division:

$$\begin{array}{r} x-5 \\ x-1\,\overline{)\,x^2-6x+5} \\ -(x^2-x) \\ \hline -5x+5 \\ -(-5x+5) \\ \hline 0 \end{array}$$

0 ← remainder.

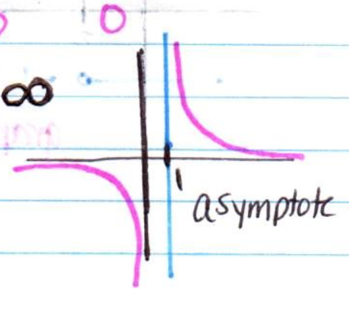
$x\cdot x = x^2$ | $x^2 - x^2 = 0$ | $-6x$, $-(-x) = +x$, $-6x + x = -5x$

$x\cdot -1 = -x$

$-5\cdot x = -5x$ | $-5x$, $-(-5x) = +5x$, $-5x+5x = 0$ | $5 - 5 = 0$

$-5\cdot -1 = 5$

$\therefore \lim\limits_{x\to 1} = \frac{(\cancel{x-1})(x-5)}{(\cancel{x-1})(x-1)} = \lim\limits_{x\to 1}\frac{x-5}{x-1} = \lim\limits_{x\to 1}\frac{1-5}{1-1} = \frac{-4}{0} = -\infty$



Absolute value functions

if $\lim_{(x \to a)} f(x) = f(a)$

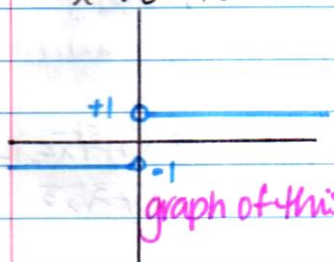
$$\lim_{x \to a} |f(x)| = |f(a)| = \left|\lim_{x \to a} f(x)\right|$$

Eg: $\lim_{x \to -2} |x^2 - 2x + \sqrt[3]{2}| \Rightarrow \lim_{x \to -2} |-2^2 - 2(-2) + \sqrt[3]{2}| = |8 + \sqrt[3]{2}|$

$4 + 4 + \sqrt[3]{2}$

Sometimes absolute values exist inside other functions

Eg $\lim_{x \to 0} \frac{|x|}{x} \Rightarrow \frac{0}{0}$ $|x|$ is like a piecewise fx.



$$|x| = \begin{cases} (x) & x \geq 0 \\ -(x) & x < 0 \end{cases} \rightarrow \text{eg. } -2 = x < 0$$

$-(-2) = +2 = |x|$

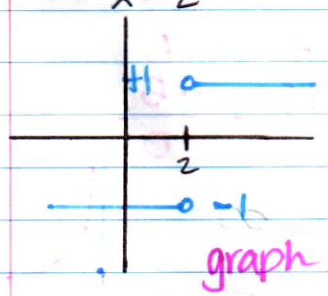
So we have to check the limits on the (L) and (R) sides of this function because 0 is a boundary point.

$\lim_{x \to 0^-} \frac{|x|}{x}$ ($x < 0$ so we use bottom fx) $\Rightarrow \lim_{x \to 0^-} \frac{-x}{x} = \frac{-1}{1} = -1$

$\lim_{x \to 0^+} \frac{|x|}{x}$ $(x \geq 0) \therefore \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} \frac{1}{1} = 1$

$1 \neq -1$; $\lim_{x \to 0^-} \neq \lim_{x \to 0^+}$ $\therefore$ the limit does not exist.

Eg $\lim_{x \to 2} \frac{|x-2|}{x-2} \Rightarrow \frac{0}{0}$ $|x-2| = \begin{cases} x-2 & x \geq 2 \\ -(x-2) & x < 2 \end{cases}$ } break it into a piecewise fx.



$\lim_{x \to 2^-} \frac{-(x-2)}{x-2} \Rightarrow \lim_{x \to 2^-} \frac{-\cancel{(x-2)}}{\cancel{(x-2)}} \Rightarrow \frac{-1}{1} = -1$

$\lim_{x \to 2^+} \frac{\cancel{(x-2)}}{\cancel{(x-2)}} \Rightarrow \lim_{x \to 2} \frac{1}{1} = 1$

$-1 \neq 1$ $\therefore$ limit $\nexists$

$\lim_{x \to 2^-} \neq \lim_{x \to 2^+}$

Parametric or piecewise functions

$$f(x)\begin{cases}\boxed{\sqrt{x+1}} & x \geq 0 \\ \boxed{d+x} & x<0\end{cases}$$

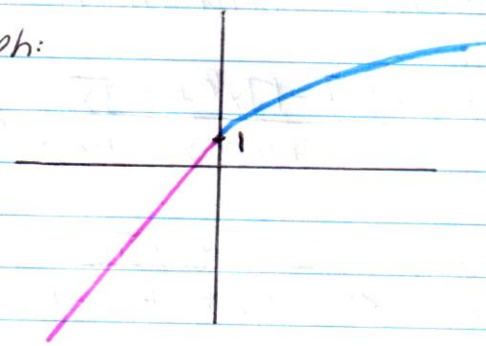
Q = what is the value of d so that f(x) is continuous @ all points in its domain?

Dom $f(x) = x \in \mathbb{R}$

The problem with parametric functions is that the Ⓛ+Ⓡ limits may not agree at their boundary point. To make f(x) continuous we need to find a value for d that makes the Ⓛ=Ⓡ limit @ boundary point 0. (for all other values, f(x) is continuous.)

$\lim\limits_{x\to 0^{+}} = \boxed{\sqrt{1+0}} = \sqrt{1} = 1$ for f(x) to be continuous, the Ⓛ limit must = 1

$\lim\limits_{x\to 0^{-}} = \boxed{d+0} = 1 \Rightarrow d = 1$ this is the graph:



$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = \lim_{x=0} f(0)$$
$$1 = d = 1 = 1$$

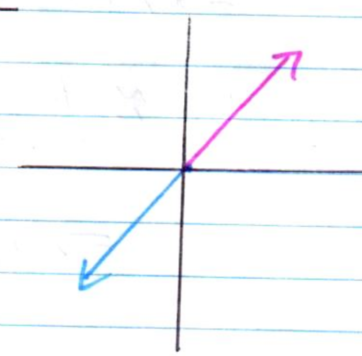
★ Check Ⓛ limit & Ⓡ limit @ all boundary points.

Infinite limits:

Eg: $\boxed{\lim\limits_{x\to\infty^{\pm}} 25 = 25} \Rightarrow \lim\limits_{x\to\infty^{\pm}} c = c$

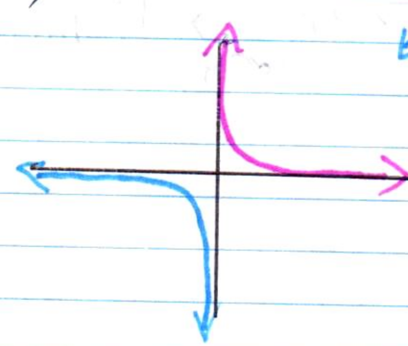
$\boxed{\lim\limits_{x\to\infty^+} x}$ $(x+c) \Rightarrow \lim\limits_{x\to\infty} x$ $(+\lim c) = +\infty$

$\boxed{\lim\limits_{x\to\infty^-} x}$ $(x+c) \Rightarrow \lim\limits_{x\to\infty^-} x$ $(+\lim c) = -\infty$



$\lim\limits_{x\to\infty} \boxed{\frac{1}{x}}$ $\left(\frac{c}{x^n}\right) \Rightarrow \lim\limits_{x\to\infty^+} = 0^+$

$\lim\limits_{x\to-\infty} \boxed{\frac{1}{x}}$ $\left(\frac{c}{x^n}\right) \Rightarrow 0^-$



eg. $f(x) = \frac{4x^5+3x^2+2}{6x^5+3x^2+12}$ $\lim_{x\to\infty^-} f(x) = ?$

Think about when these numbers get very very large.
x^2 is big but no matter what x^5 will be $> x^2$ as $x \to \infty$
So care about the term with the highest degree. ★
This f(x) will approximate (as $x \to \infty$:)

$\lim_{x\to\infty^-} f(x) \frac{4x^5}{6x^5} \approx \frac{2}{3}$ $\lim_{x\to\infty^+} f(x) \approx \frac{2}{3}$ $\approx$ = approximately equals

think about which terms will dominate the rest.

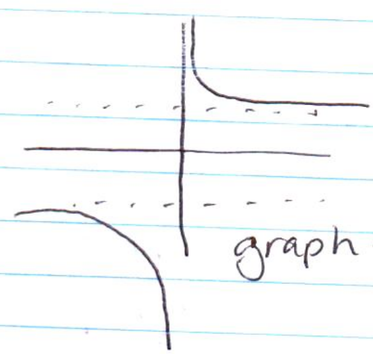
Eg: $\frac{9x^7-17x^6+15\sqrt{x}}{3x^7+1000x^5+\log_2 x}$ $\lim_{x\to\infty} \approx \frac{9x^7}{3x^7} \approx \lim_{x\to\infty} 3 = \boxed{3}$

$\lim_{x\to\infty} \frac{3x^3-2x^2+7}{6x^4-x^3+2x-100} \approx \lim_{x\to\infty^-} \approx \frac{3x^3}{6x^4} = \lim_{x\to\infty^-} \frac{1}{2x} = \boxed{0}$

$\lim_{x\to\infty} \frac{4x^4-3x^3+7x-10}{250x^3+5x-x+1000} \approx \lim_{x\to\infty} \approx \frac{4x^4}{250x^3} = \lim_{x\to\infty} \frac{4x}{250} \Rightarrow \frac{4}{250} \lim_{x\to\infty} x = \infty^+$

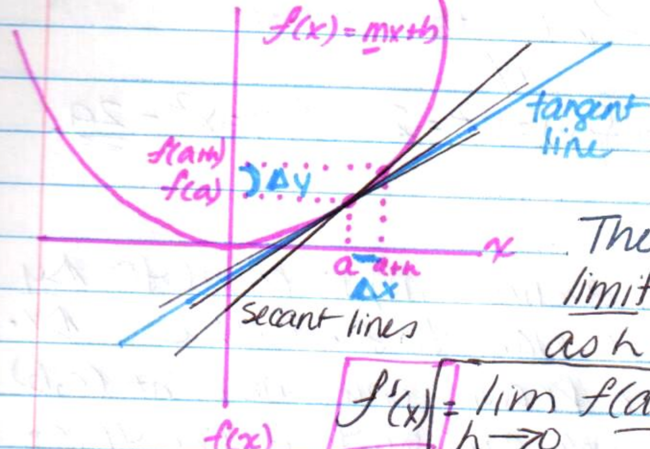
$\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} \approx \lim_{x\to\infty} \frac{1}{\sqrt{x^2}} \approx \lim_{x\to\infty} \frac{1}{|x|}$ for $x \to \infty^+$ or $x \to \infty^-$

$\Rightarrow \lim_{x\to\infty^+} \frac{x}{|x|} = 1$; $\lim_{x\to\infty^-} \frac{x}{|x|} = \frac{-1}{1} = -1$



graph.

Derivatives as the slope of a tangent line



$m = \dfrac{\Delta y}{\Delta x}$ $\quad m = \dfrac{f(a+h) - f(a)}{h}$

The tangent line's slope is the limit of the slope of the secant lines as h tends to zero.

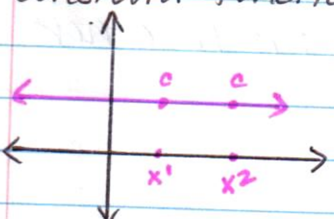
$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

The tangent line of a curve at any point is unique.

If this limit exists, then it is called the precise slope of the tangent line & we denote it by $f'(x)$, or "f prime of x." This means "the derivative of f @ $x=a$." This equation is called the first principle definition of derivatives.

Eg: simple derivatives of various continuous functions:

constant function:



$\dfrac{c - c}{x_2 - x_1} = \dfrac{0}{\Delta x} = \underline{0}$ $\quad f(x) = c$

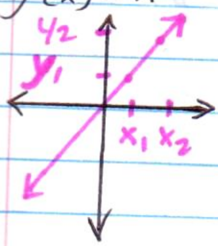
the slope of the line of constant functions is always $\underline{0}$.

$f'(x) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \dfrac{c - c}{h} = \lim_{h \to 0} \dfrac{0}{h} = 0$

identity function:

$f(x) = x$

$\therefore f'(c) = 0 \ \forall x \in \mathbb{R}$.

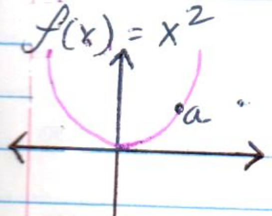


Let's pick a point. $x_1 = 1$.

$f'(1) = \lim_{x \to 1} f(x) \dfrac{1 - h - 1}{h} = \dfrac{h}{h} = 1$

the $f'(x)$ where $f(x) = x$ is always $= 1$. $\therefore f'(x) = 1 \ \forall x \in \mathbb{R}$.

$f(x) = x^2$



find the equation of the tangent line at $x = a$.

$\lim_{h \to 0} f(a) = \dfrac{f(a+h) - f(a)}{h} = \lim_{h \to 0} f(a) = \dfrac{(a+h)^2 - a^2}{h} =$

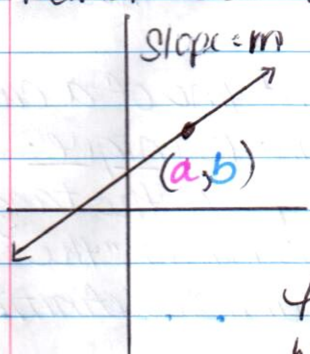
$= \lim_{h \to 0} f(a) = \dfrac{a^2 - a^2}{h} = \dfrac{0}{0} =$ indeterminate case so let's transform the function using difference of squares:

$$\frac{(a+h)^2-a^2}{h} \Rightarrow \frac{(\cancel{(a+h)-a})((a+h)+a)}{h} = \frac{\cancel{h}(2a+h)}{\cancel{h}} = 2a+h$$

$\lim\limits_{h\to 0} f(a) = 2a+h = \boxed{2a.}$ $\therefore$ $f'(x)$ at the point a when $f(a)=x^2 = \underline{\underline{2a}}$.

Writing the slope of a line in different forms

Remember that the slope of a line (m) is equal to $\frac{\Delta y}{\Delta x}$.



So for this line we want to find the equation of the line knowing only one point (a,b) and the slope (m.) we can write it like this:

$\frac{y-b}{x-a} = m$ } this is an equation that describes this line! but this isn't any form that we recognize. Let's convert it into points that we will recognize.

$\boxed{y-b = m(x-a)}$ ← this is called "point-slope form".

imagine this point is $(7,5)$. Then we would write it as: $y-5=2(x-7)$ if slope $=2$. We can also convert it to other forms too:

$$y-5=2(x-7)$$
$$y-5=2x-14$$
$$y=2x-14+5$$
$$\boxed{y=2x-9}$$

This is y-intercept form.

If the tangent line of x^2 is always $2a$ at $x=a$ then: $m=f'(a)=2a$ for $f(x)=x^2$.

eq^n tangent line @ $x=4$: $y-f(4)=m(x-4)$

$m=2a=2\cdot 4=8$ →

$$y-4^2=m(x-4)$$
$$y-16=8(x-4)$$
$$y-16=8x-32$$
$$\boxed{y=8x-16.}$$

What is $f'x$ for $f(x)=x^3$? at $x=a$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{(a+h)^3-a^3}{h}$$

the formula for difference of cubes is $a^3-b^3=(a-b)(a^2+ab+b^2)$

$f(x)=x^3$

$$\lim_{h\to0} \frac{((a+h)-(a))((a+h)^2+a(a+h)+a^2)}{h} = \frac{\cancel{h}((a+h)^2+a(a+h)+a^2)}{\cancel{h}} = ((a+h)^2+a(a+h)+a^2)$$

$$=\lim_{h\to0} 3a^2+\underset{\cancel{2a\cdot0}}{2ah} \text{ for } h\to0 = \boxed{3a^2}$$

∴ the tangent line for the function x^3 @ the point $a = 3a^2$.
The derivative of x^3 is $3a^2$.

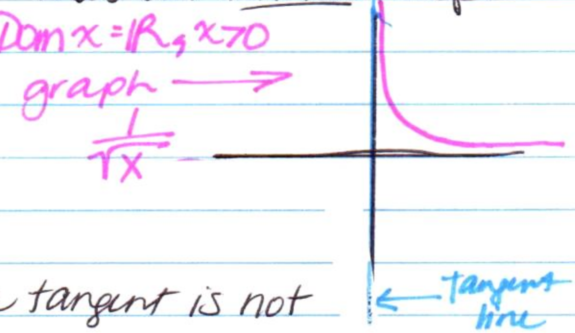
$f(x)=\sqrt{x}$. What is the derivative/the slope of the tangent line for this fx?

$$\lim_{h\to0}\frac{f(a+h)-f(a)}{h} = \frac{\sqrt{(a+h)}-\sqrt{a}}{h} = \frac{0}{0}$$

$$\lim_{h\to0} \frac{\sqrt{(a+h)}-\sqrt{a}}{h}\cdot\frac{\sqrt{(a+h)}+\sqrt{a}}{\sqrt{(a+h)}+\sqrt{a}} = \lim_{h\to0}\frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})} = \frac{h}{h(\sqrt{a+h}+\sqrt{a})} = \frac{1}{(\sqrt{a+h}+\sqrt{a})}$$

if $a=0$ and $h\to0$ then $\lim_{h\to0} = \frac{1}{0} = \infty$ ∴ the tangent line is vertical & has an infinite slope.

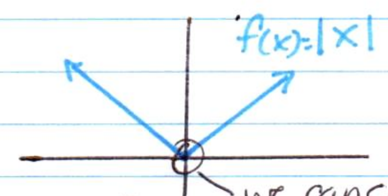
the limit of this function is unbounded, @0 therefore we say the function is not differentiable at this point.



Whenever $\underset{(L)}{\lim} \neq \underset{(R)}{\lim}$ the function is non-differentiable. The slope of the tangent is not a derivative of f'a.

Another example of a non-differentiable function:

$$f(x)=|x| = \begin{cases} (x) & x\geq0 \\ -(x) & x<0 \end{cases}$$



we can see $\lim_{x\to0^-} \neq \lim_{x\to0^+}$

because this fx's limits at the point 0 do not equal each other, $\lim_{x\to0}|x|$ ∄.

This fx is therefore non-differentiable @ $x=0$.
In general, when you see angles on a graph, the fx is non-differentia at that point.