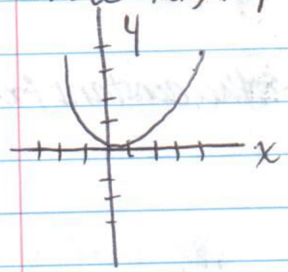
Graphing functions

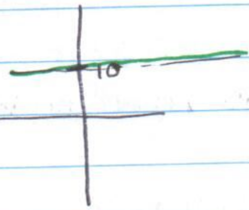
$f(x)$ are usually graphed with cartesian coordinates as $(x, f(x))$
where $f(x)$ is plotted on the y axis



$f(x) = y,\ x^2$
denotes a set of pairs with a relationship between x and y
if $f(x) = x^2$

x	y	x	y
1 =	1	-1 =	1
2 =	4	-2 =	4
3 =	9	-3 =	9

this is a symmetrical function b/c the y values are the same for +ve & -ve values of x
this is a parabolic function



$f(x) = 10$ = a constant function
for all values of x, $y = 10$
where $Dom(x) = \mathbb{R}$

$f(x) = 2x$ — a "linear" polynomial

x = y	x = y
1 = 2	-1 = -2
2 = 4	-2 = -4
3 = 6	-3 = -6

Finding the slope of the line

1) find 2 points on the line & note their coordinates. Label them 1 & 2
2) Subtract y_1 from y_2, divide by $x_1 - x_2$ (called Δy & Δx)

$m = \dfrac{y_1 - y_2}{x_1 - x_2}$ let pt 1 = (2,5), point 2 = (0,2) $m = \dfrac{5-2}{2-0} = \left.\dfrac{3}{2}\right\}$ this is the slope of the line that goes through point 1 & 2.

"m" is usually the variable assigned to the slope of the line.

Writing an equation for a line from 2 points / the slope

We know m for the previous example. If we only have one set of points and the slope, we can write the equation for the line.

$m = 3/2$

point 1 = 2,5

$y - y_1 = m(x - x_1) \Leftrightarrow y - 5 = 3/2(x - 2)$

$y - 5 = 3/2 x - 3$

$y = 3/2(x) - 3 - (-5)$

$y = 3/2(x) - 3 + 5$

this is the equation for the line $\rightarrow$ $y = 3/2(x) + 2$

Average rate of change

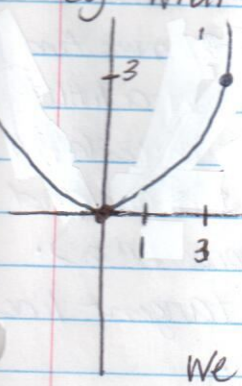
The average R.O.C. is the average of the change over a specific period

Eg: with population average R.O.C. is increase/decrease over time:

$\frac{\Delta \text{population}}{\Delta \text{time}}$ } this is the equation for average ROC over time. population is the dependent variable and time is the independent variable. So we can rewrite it in the general case like this:

$\frac{\Delta \text{dependent variable}}{\Delta \text{independent "}}$ or as we use for notation in mathematics: $\frac{\Delta y}{\Delta x} \Rightarrow \frac{\Delta f(x)}{\Delta x}$

eg: "What is the rate of change over points [1,3]? $f(x) = x^2$



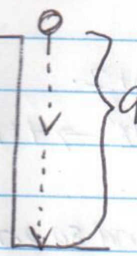
We could graph the function but we can also do it mathematically. We are given 2 values for x and a function. If $\Delta y = \Delta f(x)$ we can calculate the y values at $x=1$ & $x=3$ using the function given.

$f(1) = 1^3 = 1$

$f(3) = 3^3 = 9$

$$\frac{f(3)-f(1)}{3-1} = \frac{9-1}{3-1} = \frac{8}{2} = 4 = m$$

We can do this with more complicated functions too & apply it in other subjects like physics:



Eg: a ball falls from 90m above the ground. The height of the ground is a function of how long it takes to reach the ground. This relationship can be represented as $f(t) = 90 - 4.9t^2$.

a) determine the average rate of change (in this example, the average speed) between the first & third seconds of freefall.

$t = [1,3]$

$f(t) = 90 - 4.9t^2$

$$\frac{f_{(3)} - f_{(1)}}{3-1} \Rightarrow \frac{(90-4.9\cdot 3^2) - (90-4.9\cdot 1^2)}{3-1} = \frac{45.9 - 85.1}{2} = -19.6$$

if h = meters & t = seconds this would be -19.6 m/sec.

b) Determine the average rate of change over $[a, a+h]$

$t = [a, a+h]$

$f(t) = 90 - 4.9t^2$

$$\frac{\Delta f(t)}{\Delta t} \Rightarrow \frac{f(a) - f(a+h)}{a-a+h} \Rightarrow \frac{(90-4.9a^2) - (90-4.9(a+h)^2)}{h}$$

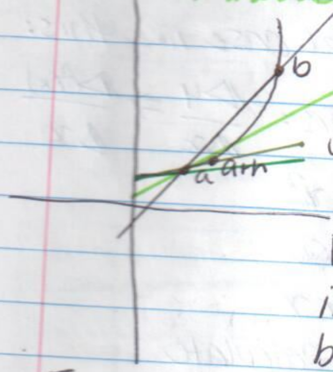
1) remove brackets by conjugating the square polynomial using $(x+y)^2 = x^2 + 2xy + y^2$

$$\frac{90 - 4.9a^2 - 90 - 4.9(a^2 + 2ah + h^2)}{h} \Rightarrow \frac{-4.9a^2 - (90 - 4.9a + 9.8ah + 4.9h^2)}{h}$$

2) flip the signs $(-(-)) = +$ $(-(+)) = -$

$$\frac{\cancel{90} - \cancel{4.9a^2} - \cancel{90} + \cancel{4.9a^2} - 9.8ah - 4.9h^2}{h} \Rightarrow \frac{-9.8a\cancel{h} - 4.9\cancel{h}^2 h}{\cancel{h}} = 9.8a - 4.9h.$$

Instantaneous rate of change



So far we have found the slope of the secant line that connects point a & b. But what if we want the slope of the tangent line that only intersects the graph at point a? We can estimate it mathematically using the previous method. We pick a second imaginary point on the line and then we find its slope. We want this point close to a so we will add a little bit to it, maybe 0.1. this is "h." This second point is called (a+h). This is still only an estimation so to get closer we can pick smaller and smaller values for h until we are really close to a (we "approach" a). It's important to note we will never find the slope of the tangent line this way - it's an estimation - but we can get pretty close.

$\frac{\Delta f(x)}{\Delta x} = \frac{f(a+h) - f(a)}{h}$ ← this is the difference equation.

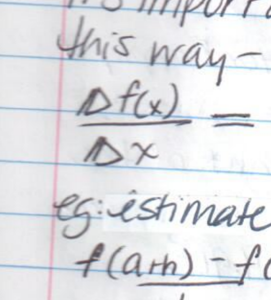
eg: estimate the instantaneous rate of Δ for $f(x) = x^2$ @ the point 2.

$$\frac{f(a+h) - f(a)}{h} \rightarrow \frac{f(2+h)^2 - f(2)^2}{h} \rightarrow \frac{(2+h)^2 - 4}{h} \rightarrow \frac{4 + 4h + h^2 - 4}{h} \rightarrow \frac{4\cancel{h} + h^{\cancel{2}}}{\cancel{h}} \rightarrow 4 + h$$

If we give h the value of 0.1 we get 0.1. 0.01 = 4.01 0.001 = 4.001 and so on. This is still not the slope of the tangent line. The tangent line is unique. The notation for the type of question: $\frac{\Delta f(x)}{\Delta x}(a)$ by h, $h \rightarrow 0$, find m

a tangent line is a kind of secant line that only intersects @ one point.

Limits



approximating the area of a circle w/ polygons

$S = S_4, + S_5 + S_6 + S_7 + S_8 \dots S_n$
(area of circle)

S = sides We would write this as $S_n \rightarrow S$ as $n \rightarrow \infty$

"S_n approaches S as n tends to infinity"

Eg: $f(x)=x^2$ What is the limit of $f(x)$ as x approaches 3?

$f(3)=3^2=9$ ← this is all well and good but unless you check on both sides of 3 you won't be really sure. Not all functions are as straight-forward as x^2.

$f(3.01)=3.01^2=9.006$ $f(2.99)=2.99^2=8.9401$

So we can say that as $x \to 3$, $f(x)$ approaches 9.

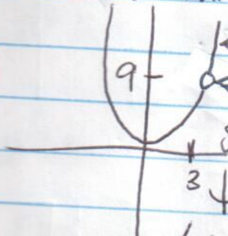
$\therefore \lim\limits_{x \to 3} x^2 = 9$

the bottom is very important. Without $x \to 3$, this limit has NO MEANING!

this function is also <u>continuous</u> @ the point 3. It has no missing points.

In fact this function is continuous @ all points in its domain.

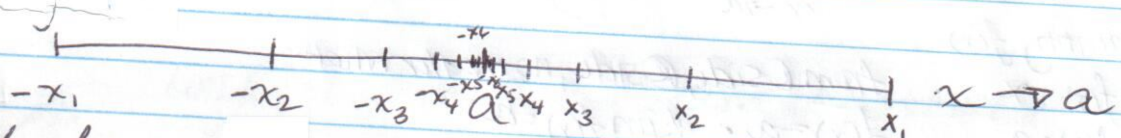
What if a function is <u>not</u> continuous?



$f(x)=x^2, x \in \mathbb{R}, x \neq 3.$

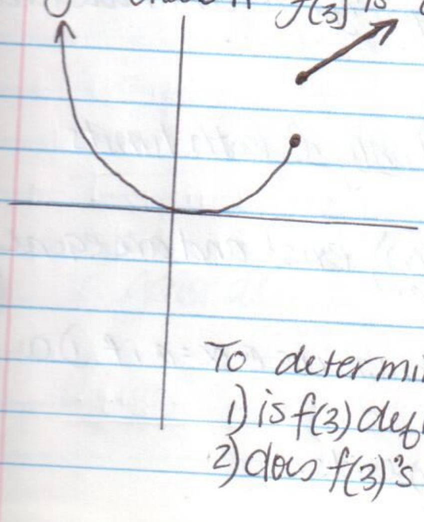
say we removed a single point in this $f(x)$ If we <u>removed</u> point 3, then the function would <u>not</u> be continuous @ point 3.

$\lim\limits_{x \to 3} (x^2; x \in \mathbb{R}, x \neq 3) \neq f(3)$



In general, f is continuous @ $x=a$ in its domain if $\lim\limits_{(a)} f = f(a)$

eg: "check if $f(3)$ is continuous."



$$f(x) \begin{cases} x^2 & x<3 \\ x+8 & x \geq 3 \end{cases}$$

for (3), all $f(3)$ will be <9
(cannot approach 3 from its ® side)
all $f(3)$ will be ≥ 11
(cannot approach 3 from its ® side.)

To determine if this $f(x)$ is continuous we must answer 2 questions:
1) is $f(3)$ defined in the $f(x)$? — yes
2) does $f(3)$'s limit exist?

$f(3) = 11$

(from the Ⓛ side) $\lim_{x \to 3^-} f(x) = 9$

(from the Ⓡ side) $\lim_{x \to 3^+} f(x) = 11$

because this function has >1 limit, there is no unique, fixed value for this function's limit. The limit must be a unique fixed value in order to exist ∴ the limit for this function does not exist.

$\lim_{x \to 3} \neq f(3)$

functions are discontinuous if either:

a) The limit is at the point a but there is discontinuity @ point a the limit = a value but is not equal to f(a). (previous example)

b) the limit does not exist for the function @ $x \to a$.

Examples of continuous functions:

constant f(x)

1) $f(x) = c$
c = constant

$f(x) \to c$
$\lim_{x \to a} f(x) = c$

} continuous @ all points in its domain

identity f(x)

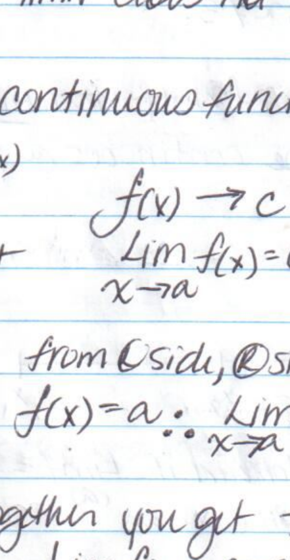
2) $f(x) = x$
$f(a) = a$

from Ⓛ side, Ⓡ side, no matter what $f(x) = a$ ∴ $\lim_{x \to a} f(x) = a$

3) add 1) + 2) together you get $f(x) = a + c$ ◁— add any two continuous f(x) together and you get a continuous function

$\lim_{x \to a} f(x) = a + c$

Eg: $f(x) = x^3$
$a = -2$
$f(a) = -8$



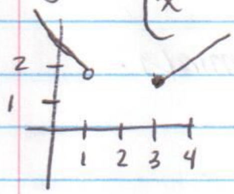
$\lim_{x \to a} f(x)$ exists if and only if both limits $\left(\lim_{x \to a^-} f(x)\right)$ and $\left(\lim_{x \to a^+} f(x)\right)$ exist and are equal to $\lim_{x \to a} f(x)$. f(x) is continuous @ x = a if 1) a exists and 2) $\lim_{x \to a} f(x)$ exists.

B) $f(x) = x^3$ for $x \neq -2$ ie $Dom(f) = \mathbb{R} \setminus \{-2\}$

$\lim_{x \to -2^-} f(x) = -8 = \lim_{x \to -2^+} f(x)$

so $\lim_{x \to -2} f(x)$ exists but -2 $\nexists$ so this function is not continuous @ point -2

C) $f(x) = \begin{cases} x+3 & x<1 \\ x & x \geq 3 \end{cases}$ $Dom(f) = \mathbb{R} \setminus [1,3)$ (undefined for $1 \leq x < 3$)



$\lim_{x \to 1^-} f(x) = 2$ $\lim_{x \to 1^+} f(x)$ does not exist (cannot approach 1 from the (R) side)

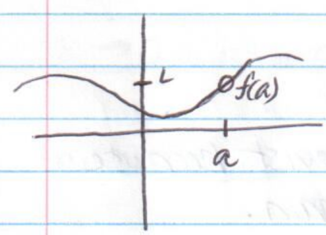
one of the limits $\nexists$ therefore $f(x)$ is not continuous @ $1 = x$

for $a=3$ $\lim_{x \to 3^-} f(x)$ $\nexists$ because we cannot approach 3 from its (L) side $\therefore$ $f(x)$ is not continuous @ 3

for $a=5$: $\lim_{x \to 5^-} f(x) = 5 = \lim_{x \to 5^+} f(x)$ both limits exist; $f(5)$ exists (is defined) so $f(x)$ is continuous @ $x=5$

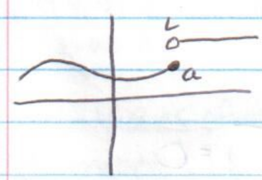
Evaluating Limits:

$\lim_{x \to a} f(x) = \lim(a)$ $\longrightarrow$ $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ $\therefore$ the limit exists and it is continuous



$\lim_{x \to a} f(x) = L, \neq f(a)$ $\therefore$ this function exists but is not continuous @ a

This is called a "removable discontinuity" because only one point is removed



$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ therefore $\lim_{x \to a} f(x)$ does not exist.

In general:

1) $\lim_{x \to a} c = c$

2) $\lim_{x \to a} x = x$

3) if $\lim_{x \to a} f(x) = f(a)$ & $\lim_{x \to a} g(x) = g(a)$

then

$\lim_{x \to a} (f(x) \pm_{\times} g(x)) = f(a) \pm_{\times} g(a)$

This works for limits that are continuous functions like polynomials.

$\lim a_n b^n + a_{n-1} b^{n-1} + a_{n-2} b^{n-2} + etc\ldots = a_n b^n + a_{n-1} b^{n-1} + a_{n-2} b^{n-2} \ldots$

Eg: $\lim_{x \to 2} 5x^6 + \sqrt{2x^2} + 1 = \lim_{x \to 2} 5x^6 + \lim_{x \to 2} \sqrt{2x}^2 + \lim_{x \to 2} 1$

4) $\lim_{x \to a} f(x) = f(a) \Rightarrow \lim_{x \to a} \sqrt[n]{f(x)} = \lim_{x \to a} \sqrt[n]{f(a)}$ provided $\sqrt[n]{f(a)} = \mathbb{R}$.

Remember, for radicals of even degrees, what's under can't be -ve.
$\sqrt{-1}$ is not a real number. $\sqrt[3]{-1}$ is a real number.

$f(x)\sqrt{x} \Rightarrow \lim_{x \to 0^+} \sqrt{x} = \sqrt{\lim_{x \to 0^+} x} = \sqrt{0} = 0$

Dom $f(x)$ $[0, +\infty)$. The (L) sided limit $(x \to 0^-)$ does not exist because $x<0$ is not defined in this function. It is not in its domain.
$\lim_{x \to 0^-} \sqrt{x}$ does not exist.

This is only for radicals of even degrees.
Eg: $\lim_{x \to 1} \sqrt[3]{x^2 - 2x + 1} \rightarrow \sqrt[3]{1^2 - 2 + 1} = \sqrt[3]{-1+1} = \sqrt[3]{0} = 0$

both $\lim_{x \to 1^-} (x)$ and $\lim_{x \to 1^+} x$ exist and are equal (0.)

5) if $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$

then $\dfrac{\lim f(x)}{\lim_{x\to a} g(x)} =$

- $g(a) \neq 0 \Longrightarrow \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{f(a)}{g(a)}$
- $g(a) = 0$
  - $f(a) \neq 0 \Rightarrow \lim_{x\to a} \dfrac{f(x)}{g(x)} \to \dfrac{\lim f(a)}{\lim g(a)} = \dfrac{\mathbb{R}}{0} \to \infty$
    or: does not exist
  - $f(a) = 0 \to \lim_{x\to a} \dfrac{f(a)}{g(a)} = \dfrac{0}{0}$ = indeterminate limit

with division, check the domain of the function first because the limit will be different based on whether or not the denominator $= \emptyset$

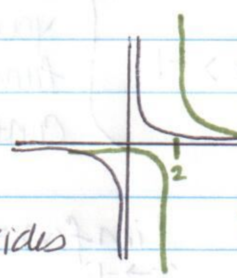
eg: $\lim_{x\to 2} \dfrac{\sqrt{2}x - 2\sqrt{2}}{x^2-1} = \dfrac{\sqrt{2}2 - 2\sqrt{2}}{2^2-1} = \dfrac{0}{3} = 0$

$x \to 0$ is not zero! If $\dfrac{\mathbb{R}}{x\to 0}$ then the denominator will get smaller and smaller (not zero)

and the value of the radical will get bigger and bigger $\longrightarrow \infty$

eg: $\lim_{x\to 0} \dfrac{1}{x} = \dfrac{\lim 1}{\lim x} \begin{matrix}\longrightarrow \\ \longrightarrow\end{matrix} \dfrac{1}{0^+} \longrightarrow \infty$ ← not exactly zero

The y axis of the graph of this function is an asymptote line of the graph of $\frac{1}{x}$ on both sides



$\lim_{x\to 2} \dfrac{x^2+1}{x-2} = \dfrac{4+1}{0}$ $\dfrac{5}{0}$

We don't need to estimate when we have continuous polynomials. The limit = the function when you substitute a in the variable of the function. For any polynomial you can substitute it in to find the limit of the polynomial.

when the limit is indeterminate, try to make it determinate.

Eg: $\lim_{x\to 1} \frac{x-1}{x^2-1} \begin{matrix}\to 0\\ \to 0\end{matrix}$ = indeterminate $\to \frac{(\cancel{x-1})}{(\cancel{x-1})(x+1)} \to \frac{1}{x+1}$

if we factor out the denominator's polynomial we can cancel them & make the numerator 1. $\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$

we have determined the limit and made it determinate.

Eg: $\lim_{x\to 25} \frac{5-\sqrt{x}}{x-25} = \frac{5-5}{25-25} \to \frac{0}{0}$ $\quad x \to a$, $x-a \to \emptyset$

find x-a. x-25 exists in the numerator, if we rationalize:

$$\lim_{x\to 25} \frac{5-\sqrt{x}}{x-25} \cdot \frac{5+\sqrt{x}}{5+\sqrt{x}} = \frac{x+25}{x-25(5+\sqrt{x})} \to \frac{-1(\cancel{x-25})}{(\cancel{x-25})(5+\sqrt{x})} \to \frac{-1}{5+\sqrt{x}}$$

$$\lim_{x\to 3} \frac{\frac{1}{3}-\frac{1}{x}}{x-3} \to \frac{\frac{1}{3}-\frac{1}{3}}{3-3} \to \frac{0}{0}$$

$$\to \frac{\frac{1}{3}-\frac{1}{x}}{x-3} \quad \frac{\frac{x-3}{3x}}{x-3} \to \frac{\cancel{x-3}}{3x} \cdot \frac{1}{\cancel{x-3}} \to \frac{1}{3x} = \frac{1}{9}$$

$$f(x) \begin{cases} 2-x^2 & x \leq -1 \\ x-1 & x > -1 \end{cases}$$

What about piecewise functions? If there's a rule change at the point where you want to evaluate the limit you need to check the limit on both sides of the function. The cases described above only work on continuous functions.

Check $\lim_{x\to 1}$.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 2-x^2 \to 2--1^2 \to 2-1 \to 1$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x-1 \to -1-1 \to -2 \neq 1$$

$\therefore$ this f(x) is NOT continuous.