Sep 5 MAT 1331

Marking Scheme:

Assignments:	Sept 27	5%	
	Oct 25	5%	
	Nov 8	5%	
	Nov 29	5%	
	total	20%	
Midterms	Oct 1	15%	
	Nov 27	15%	
	total	30%	
Final		50%	
	final Mark		

Lec 1: Set notation, number types & rules

Set notation

a set is a collection of elements

elements can be expressed in a roster or according to a rule

rules are expressed in notation:

$S = \{2, 3, \sqrt{2}\}$ ← "S is a set with 3 members"

$S = \emptyset = \{\ \}$ → "S is an empty set"

$S \neq \emptyset$ → "S is not an empty set"

$A = \{2\}$

$B = \{2, 1/3\}$

$2 \in S$ ← "2 is a member of set S"

$4 \notin S$ ← "4 does not belong to set S"

$A \subset S$ ← "A is a subset of S"

$B \not\subset S$ ← "B is not a subset of S"

$S \cup B = \{2, 3, \sqrt{2}, 1/3\}$ ← $S \cup B$ = "Set S union (w/) set B equals" (both sets combined)

$S \cap B = \{2\}$ ← (only what is common to both sets) "S intersect B equals"

$\mathbb{N}$ = natural numbers $(1, 2, 3, 4, \ldots \infty)$

$\mathbb{Z}$ = integers $(-\infty, \ldots -3, -2, -1, 0, 1, 2, 3 \ldots \infty)$

$\mathbb{Q}$ = all rational numbers (which can be expressed as a fraction)

$\mathbb{R} = \{\mathbb{Q} \cup \mathbb{Q}^c\}$ = all real numbers (rational #s union w/ irrational #s)

$\forall$ = "for any/all" e.g. $\forall x \in S$ = "for any values of x belonging to set S"

$\exists$ = "there exists some" e.g. $\exists x \in S$ = "there exist some values of x belonging to S"

irrational #s cannot be represented by a fraction ie. $\pi, e, \sqrt{2}, \sqrt{3} = \mathbb{Q}^c$

$\mathbb{Q} = \{\frac{m}{n} ; m, n \in \mathbb{Z}, n \neq 0\}$ = "rational #s are a set of numbers that can be represented as fractions, where both the numerator & denominator are integers ($\mathbb{Z}$) and the denominator is not zero"

= the real line

0

all real #s can be represented on this bidirectionally infinite line

Subsets of real #s

[1,10] = a subset of the real line containing all #s from 1–10

$[1,10] = \{x \in \mathbb{R};\ 1 \leq x \leq 10\}$

[1,10] =

0 1 10

[ ] = including

[ ) ← not including e.g.

(←] (↔)

$[1,10) = \{x \in \mathbb{R};\ 1 \leq x < 10\}$ (not including 1)

$(1,10] = \{x \in \mathbb{R};\ 1 < x \leq 10\}$ (not including 10)

$(1,10) = \{x \in \mathbb{R};\ 1 < x < 10\}$ (not including 1–10)

Rules

1) $\sqrt[n]{x}$ when n is an even #, $x \geq 0$

2) $x^{-n} = \frac{1}{x^n}$

3) $x^0 = 1 \ \forall x \in \mathbb{R}$

4) $x^{\frac{1}{n}} = \sqrt[n]{x}$ & $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

5) $x^n \cdot x^m = x^{n+m}$

6) $\frac{x^n}{x^m} = x^{n-m}$

7) $(xy)^n = x^n \cdot y^n$

8) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

9) $(x^n)^m = x^{n \cdot m}$

important identities for simplifying polynomials

1) $(x+y)^2 = x^2 + 2xy + y^2 = (x+y)(x+y)$

2) $(x-y)^2 = x^2 - 2xy + y^2$

3) $(x-y)(x+y) = x^2 - y^2$

4) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 + y^3$

"polynomials" = mixtures of variables w/ natural exponents & natural coefficients

variable

$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots a_1 x + a_0$ constant

degree = the biggest exponent of the function

Simplest = $P_1 x$ · aka $a_1 x + a_0$ (eg $2x+1$ or a "linear expression" "finding the root" = solve for the value(s) of x which make the $fx = 0$

polynomials of degree 2 are quadratic polynomials

$P_2x = a_2x^2 + a_1x + a_0$ or $P_2x = ax^2 + bx + c$

$P_2x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ * according to rule #1, start by calculating $\sqrt{b^2-4ac}$ to see if the result is +ve or zero *

polynomials of degree 2 have @ most 2 roots - ⊕ve and ⊖ve

if $\Delta = \sqrt{b^2-4ac}$ then $\Delta > 0 \; \exists \, x_1, x_2$ "there exists 2 roots"

$\Delta = 0 \; \exists \, x_1 = x_2$ "there exists 1 root"

$\Delta < 0 \; \nexists \, x$ "there exist no roots"

p1012 Solving inequalities

Incl. ~~Linear inequalities → polynomials of degree 1~~ ;

quadratic inequalities → polynomials of degree 2 ;

f of Real numbers. Notation: f = function

In general: eg. solving inequalities

1) start by finding the root.
   put all variables on one side

$P(x) = 2x+1 \geq 3x+5$

$P(x) = 2x - 3x \geq 5 - 1 \rightarrow -x \geq 4 \rightarrow x \leq -4$ change the direction

2) if x is any Real number, what is the sign of $P(x)$?
   to show this on a table:

take out "or is equal)

$P(x) = x + 4 \geq 0$ which implies that

the real line →

x	for all values < the root	-4 (the root)	for all values > the root
$x+4 \geq 0$	$-$	0	$+$

$P(x) = x+4 > 0 \Rightarrow x > -4$ — ⊕ve

$x+4 < 0 \Rightarrow x < -4$ — ⊖ve

In the general case:

x		$-b/a$	
$ax+b$	$-$	0	$+$
	opposite sign of coefficient a		same sign as coefficient a

The sign of the polynomial has a relationship with its coefficient a.

Quadratic polynomials

$a^2x + bx + c$ where $\Delta = b^2 - 4ac$

There are 3 cases: 1) $\Delta > 0 \rightarrow$ there are 2 roots: $x_1, x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$

2) $\Delta = 0 \rightarrow$ there is one root: $\frac{-b}{2a}$

3) $\Delta < 0 \rightarrow$ there are no $\mathbb{R}$ roots

$\Delta > 0$

1)

x	$-\infty$	x_1		x_2	$+\infty$
	the same sign as a	0	the sign opposite a	0	the same sign as a

eg. $2x^2 - x - 2 = P_{2(x)}$

① find $\Delta = b^2 - 4ac$

$\Delta = a = 2,\ b = -1\ \ c = -2$

$\Delta = -1^2 - (4 \times 2 \times -2)$

$\Delta = 1 - (-16)$ or $1 + 16$

$\Delta = 17$ which is $> 0\ \therefore$ 2 roots

$\leftarrow$ so we use the quadratic eq'n

$x = \frac{-b \pm \sqrt{\Delta}}{2a} \rightarrow x_1 = \frac{1 - \sqrt{17}}{4}$

$\rightarrow x_2 = \frac{1 + \sqrt{17}}{4} \rightarrow$

a is ⊕ve $\therefore$

x		$\frac{1-\sqrt{17}}{4}$		$\frac{1+\sqrt{17}}{4}$	
	+	0	−	0	+

2) $\Delta = 0 \Rightarrow x_1 = x_2$

e.g. $x^2 - 4x + 4$

$\Delta = -4^2 - (4)(4)(1)$

$\Delta = -4^2 - (-16)$

$\Delta = -16 - (-16)$

$\Delta = -16 + 16 = 0 \rightarrow$ 1 root (case 2)

x		$x_1 = x_2$	
$P_{2(x)}$	⊕ or ⊖	0	⊕ or ⊖

always the same sign as (a)

3) $\Delta < 0 \Rightarrow$ 0 $\mathbb{R}$ root

the expression will not = 0 for any value of x

the polynomial will have the same sign as (a)

x	$\overbrace{\qquad\qquad}^{\text{no } \mathbb{R} \text{ root}}$
$P_{2(x)}$	⊕ or ⊖ the same sign as a

e.g. $x^2 - x + 1$

$\Delta = +1^2 - (-1)(4)(1)$

$\Delta = 1 - 4$

$\Delta = -3 < 0\ \therefore$ no $\mathbb{R}$ root (case 3)

so just look @ a

if a is ⊕ for all values of x, $P_{2(x)}$ will be +

What if $P_{(n)}(x)$ has 3 degrees or more?
↳ factor it out into a polynomial that is a <u>product</u> of polynomials of degree 1 or 2

e.g. $\underbrace{x^3 - 5x^2}_{x^2 = \text{common factor}} \underbrace{- 5x - 25}_{5 = \text{common factor}}$

↳ $x^2(x-5) - 5(x-5)$ — common factors

↳ $(x^2-5)(x-5)$

we can simplify this using polynomial identity 3) $(x^2-y^2) = (x+y)(x-y)$

↳ $5 = (\sqrt{5})^2$

↳ $(x+\sqrt{5})(x-\sqrt{5})(x-5)$

find the roots

$(x+\sqrt{5})(x-\sqrt{5})(x-5) = 0$

any one of these linear polynomials must $= \varnothing$ for this expression to be $=$ to 0

therefore: (or $\Rightarrow$)
$$\left\{\begin{array}{ll} x-5=0 & x=5 \\ x-\sqrt{5}=0 \Rightarrow & x=\sqrt{5} \\ x+\sqrt{5}=0 & x=-\sqrt{5} \end{array}\right\}$$

graph the zeros on a table ⟶

x		$-\sqrt{5}$		$\sqrt{5}$		5	
$x-5$	−		−		−	0	+
$x-\sqrt{5}$	−		−	0	+		+
$x+\sqrt{5}$	−	0	+		+		+
$P_3(x)$	−		+		−		+

remembering the <u>general case</u> for linear polynomials we mark whether the sign of the polynomial is + or −

$P_3(x)$ depends on the signs of each linear polynomial, so multiply the signs together
∴ this is the sign of the polynomial of degree 3

Functions

A <u>function</u> is a mathmatical relationship between two variables
↳ the independent variable, usually shown as (x)
↳ the dependent variable, usually shown as (y)

the <u>domain</u> of the function is the <u>set</u> of all possible allowed values for x

the <u>range</u> or the <u>image</u> of the function is the set of all the values for y which correspond to an allowed value for x

f is the notation used to denote a function

in other words, a f is a relationship between two sets.

Eg: Set A : Set B

the set of : the set of $f(a)=b$ or $f(x)=y$

independent : dependent "members of set b are a function of set a"

variables : variables "members of set (y) are a function of set (x)"

a ⟶ b are dependent to

"parents" "children"

a depends on b

parents can have more than one child (dependent)

but children may not be associated w/ more than one couple.

What is a domain?

All elements of set x which can participate in the function so that y is a Real number.

"what is the domain" = "what are the valid values of x?"

Eg: price & demand x = the independent variable, demand

$\underbrace{P(x) = 120 + 0.3x}_{\text{this is the function}}$ $P(x)$ = the dependent variable, price

We will say that 120 is the minimum price.

What is the domain of the f? (valid values for x)

If 120 is the minimum price, then $0.3x$ cannot be less than 0

↳ x cannot be < 0

$\therefore \text{Dom}_{(P_x)} = \mathbb{R}; \forall x, x \geq 0$

$\text{Domain}_{(P_x)}$ = all Real #s; $\forall x, x \geq 0$

where for all values of x, x is greater than or = to zero

$\text{Image}_{(P_x)} = P(x) \geq 120$ On the real line:

x: 0

$P(x)$ = 120

In general, the domain of all polynomials follows 2 rules

1) dividing by zero is not a $\mathbb{R}$
2) for $\sqrt[n]{m}$ if ^{n} is even, m must be ⊕

functions are relationships from $\mathbb{R} \to \mathbb{R}$
so from one set of $\mathbb{R}$ to another set of $\mathbb{R}$
$f = a \to b$
for some points in set a we associate a unique element from set b
but 2 elements in a may be related to 1 element in B.
The simplest example of a function is where y = a constant $\mathbb{R}$ number

$f(x) = c$

Dom $(f(x)) = \mathbb{R}$ $\quad$ Im $(f(x)) = \{c\}$

or: the identity function
where $x = y$
$f(x) = y$ $\quad$ Dom$(f_x) = \mathbb{R}$
Im $(f_x) = \mathbb{R}$
the range/image is the same as the domain because x associates to itself

A domain is all values of x where y would equal any Real number.
A domain is sometimes called the "preimage."

eg: $y = x^2$

set of x → set of y
$0 \longrightarrow 0$
$1 \longrightarrow 1$
$2 \longrightarrow 4$
$3 \longrightarrow 9$
....
$-1 \longrightarrow 1$
$-2 \longrightarrow 4$

"preimage" (domain) $\quad$ image (range)

Dom $f(x) = \mathbb{R}$
Im $f(x) = \{y \in \mathbb{R} : y \geq 0\}$

where y is a Real number

because $-x^2 = y$
$\therefore y$ is ⊕

Remember the polynomial function?

$f(x) = Pn(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + a_{n-3} x^{n-3} + \dots a_1 x + a_0$

a_0 is a constant

In $Pn(x)$ if $n=0$ then $Pn(x) = a_0$ = a constant

∴ a constant is a polynomial of degree zero

the identity function is a f of degree one (a_i)

identity $f = a_1 x + a_0$ or $x + 0$

degree!

$\forall i,\ 1 \leq i \leq n \quad a_i = 0 \quad a_0 = c$ (constant)

for all values of

(because if i was < 1 it would be a_0 since degrees are Natural numbers)

Fractional functions or "rational functions"

$\mathbb{Q}$ = the set of rational numbers

e.g: $f(x) = \dfrac{Pn(x)}{Pm(x)}$ 2 polynomials $\mathbb{Q}fx$ = rational function

$f(x) = \dfrac{x+1}{x^3-2x} = \mathbb{Q}f(x)$

not all rational fx are polynomials. But all polynomials are $\mathbb{Q}f(x)$

$f(x) = \dfrac{Pn(x)}{1} = Pn(x) = \mathbb{Q}fx.$

domains for $\mathbb{Q}fx$:

1) the denominator cannot be equal to zero!
2) $\sqrt[n]{m}$ → if n is an even number, m must be ⊕ve

because we are relating $\mathbb{R}$ to $\mathbb{R}$

e.g: $f(x)\ \dfrac{x+1}{x^3-2x}$ ← what values are equal to zero? The roots of the polynomial in the denominator!

1) factor into a product of polynomials of degree 1 or 2:

$x^3 - 2x = x(x^2-2)$

2) find the roots:

$x(x^2-2) = 0 \therefore x = 0$ or $x^2 - 2 = 0 \Rightarrow \pm\sqrt{2} = x$

it follows that

for $x^2 - 2 = 0$, x^2 would have to $= \pm 2$

therefore $\operatorname{dom} f(x) = \{x \in \mathbb{R};\ \mathbb{R} - 0,\ \mathbb{R} - \pm\sqrt{2}\} \therefore x = \pm\sqrt{2}$

Irrational $f(x)$ = $f(x) = x^{\frac{m}{n}}$ where $n \neq 1$

$= \sqrt[n]{x^m}$

$= Qc$ (irrational) fx

If n is an even number then $x^m \geq 0$

- if n is odd then find the domain of x^m
- if n is even, find the values of x^m that are ⊖ve and remove them from the set

eg: $f(x) = \sqrt[2]{x}$ $\quad$ dom $f(x) = \{x \in \mathbb{R} ; x \geq 0\}$ ← Sometimes written as $\mathbb{R}^{\oplus}$

$\mathbb{R}^{\oplus}$ = all ⊕ve $\mathbb{R}$ including 0

$[0, \infty) = \mathbb{R}^{\oplus}$

note: we never use a square bracket to bound ∞ because ∞ is not a $\mathbb{R}$.

$\mathbb{R} = (-\infty, +\infty)$

eg: $f(x) = \sqrt[3]{\frac{x}{x-1}}$ $\quad$ 1) n is odd, so we find the domain of what's under the radical.

$\frac{x}{x-1}$ ← can't $= \emptyset$ $\therefore x \neq 1$

$\text{Dom}\left(\frac{x}{x-1}\right) = \{x : x \in \mathbb{R} ; x \neq 1\}$

or we can write shorthand: $\text{Dom}(f) = \{\mathbb{R} - 1\}$

eg: $f(x) = \frac{x}{\sqrt[8]{x^2-2}}$ $\quad$ 8 is even, so find the sign of $x^2 - 2$

$x^2 - 2 = 0$ find the roots

$x^2 = 2$

$\pm\sqrt{2} = 0$

x		$-\sqrt{2}$		$\sqrt{2}$	
$x^2 - 2$	+	0	−	0	+

these are the $\mathbb{R}$ roots of $f(x)$

define the two sets: $\text{Dom} f(x) = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

shorthand ↑ $\quad$ union. this set & this set

not including the zeros

the long way ↓

$\text{Dom} f(x) = \{x \in \mathbb{R}, x < -\sqrt{2}\} \cup \{x \in \mathbb{R}, x > \sqrt{2}\}$

You need the ± because it could have been $+2^2$ or it could have been -2^2

$(\sqrt{0} = 0)$

Multiple rule functions

multiple rule functions are the union of two domains

$f(x) = \begin{cases} h(x); x \in \text{Domain}_1 \\ g(x); x \in \text{Domain}_2 \end{cases}$ think of it as two sets of rules

$\in$ is an element of

$f(x) = \begin{cases} x^2 \text{ if } x > 0 \\ \frac{1}{x} \text{ if } x < 0 \end{cases}$

so $f(2)$ would be in the 1st domain

$f(-1/3)$ would be in the 2nd domain

$f(2) = 2^2 = 4$

$f(-1/3) = \frac{1}{(-1/3)} = -3$

but if we asked "what is $f(0)$, the answer would be: $f(0)$ is undefined!

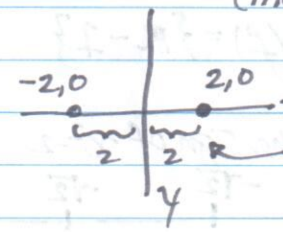
we can write the domain of these fx as: $\text{Dom} f = \{\text{Domain}_1 \cup \text{Domain}_2\}$

in this example $\text{Dom} f = \mathbb{R} - 0$ or $(-\infty, 0) \cup (0, +\infty)$

Very important special case in multiple rule functions - the absolute value fx

$fx = |x|$ 1) take a $\mathbb{R}$ 2) look @ its sign - if ⊕ve, $|x| = x$. If ⊖ve, change its sign to +ve. (multiply by -1) $|-x| = x$

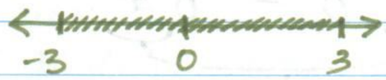
$f(x) = |x| = \begin{cases} x : x \geq 0 \\ -x : x < 0 \end{cases}$



absolute values show the distance to 0 → the distance of $\mathbb{R}$ from its origin

eg: $|x| < 3$

$0 \to 3$



$\text{Dom}_{(a)} = (-3, 3) = \{x \in \mathbb{R}; -3 < x < 3\}$

in general: $|x| < a = \{x : -a < x < a\}$ $\quad |x| > a = (-\infty, -a) \cup (a, +\infty)$

eg: $|x-1| < 2 \Rightarrow -2+1 < x < 2+1$

$\Rightarrow -1 < x < 3$

operations on sets of functions

let us imagine 2 functions: $f + g$

$(f+g)(x) = f(x) + g(x)$

$(f/g)(x) = f(x)/g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$(f-g)(x) = f(x) - g(x)$

the simplest case is the identity fx & the constant fx.

$g(x) = x$

$h(x) = 1$

$= (g+h)(x) = x+1$

$(g-h)(x) = x-1$

$(g/h)(x) = \frac{x}{-1}$

$(g \cdot h)(x) = x \cdot -1$

Composition of two functions

the notation looks like this: $(f \circ g)(x)$ — small "o"

this means find $g(x)$ then find $f(g(x))$

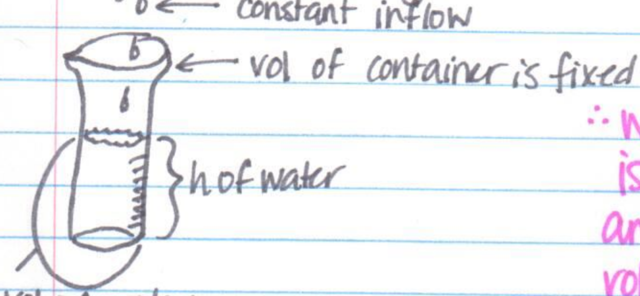
so if $f(x) = x+1$

and $g(x) = x^2$

take $g(x)$ and make it the independent variable of x

$f(g(x)) = (g(x)) + 1 = x^2 + 1$

Imagine we are filling a container with water:



The height of water is ↑ing @ a constant rate.
∴ we can say that the height of the water is a function of time. height = $h(\text{time})$
and volume is a function of height
vol = $(h(t))$.
So we write it as: $(f \circ h)(t)$