

Part A: Multiple Choice Questions (2 marks each)

- A1.** In polar coordinates, the equation of the circle $(x - 1)^2 + y^2 = 1$ is
(a) $r = \theta$, (b) $r = 1$, (c) $r = 1 + \sin \theta$, (d) $r = 2 \cos \theta$, (e) $r^2 = 1 + 2r \sin \theta$.
- A2.** The area enclosed by one loop of the lemniscate $r^2 = \sin 2\theta$ ($0 \leq \theta \leq \frac{\pi}{2}$) is
(a) $\frac{1}{4}$, (b) $\frac{3}{4}$, (c) $\frac{1}{2}$, (d) $\frac{3}{5}$, (e) 1.
- A3.** Suppose that the angle between the vectors $\mathbf{u} = \mathbf{i} - \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + a\mathbf{j}$ is $\frac{\pi}{3}$. Then a is equal to
(a) $\pm\sqrt{3}$, (b) $\frac{\pm\sqrt{3}}{2}$, (c) ± 3 , (d) ± 2 , (e) ± 3 .
- A4.** The acceleration vector $\mathbf{r}''(t)$ of $\mathbf{r}(t) = t^3 \mathbf{i} + (\sin \pi t)\mathbf{j}$ at $t = \frac{1}{3}$ is
(a) $2\mathbf{i} - \frac{\sqrt{3}}{2}\pi^2\mathbf{j}$, (b) $2\mathbf{i} + \frac{\pi^2}{2}\mathbf{j}$, (c) $3\mathbf{i} + \frac{\pi^2}{2}\mathbf{j}$, (d) $3\mathbf{i} - \frac{1}{2}\mathbf{j}$, (e) None of the above.
- A5.** Let $f(x, y) = x^3y^2$. Then $f_{xx} + f_{xy}$ is equal to
(a) $12x^2y$, (b) $3x^2 + 2y$, (c) $x^2 + y$, (d) $6x^2y + 6xy^2$, (e) $6xy + 3x^2y$.
- A6.** The tangent plane to the surface $x^2 + y^2 - 2z^2 = 3$ at the point $(2, 1, 1)$ can be written as
(a) $-2(x - 2) + 2(y - 1) - (z - 1) = 0$, (b) $2(x - 2) + (y - 1) - 4(z - 1) = 0$,
(c) $(x - 1) + 9y - 1) + (z - 1) = 0$, (d) $2x + y + z = 3$, (e) $2(x - 2) + (y - 1) - 2(z - 1) = 0$.
- A7.** Suppose that the gradient of $f(x, y)$ is given by $\nabla f = y\mathbf{i} + x\mathbf{j}$ and suppose that $w = f(s + t, s - t)$. Then $\frac{\partial w}{\partial t}$ is equal to
(a) s^2t , (b) $s - t$, (c) $-2t$, (d) $2s$, (e) $s + t$.
- A8.** Suppose $w = f(x^4 + y^2)$. Applying the chain rule twice, we have $\frac{\partial^2 w}{\partial x \partial y} =$
(a) $8x^3y f''(x^4 + y^2)$, (b) $(24x^2y + 8x^3)f'(x^4 + y^2)$, (c) $24x^2y f''(x^4 + y^2)$,
(d) $(12x^2 + 2)f''(x^4 + y^2)$, (e) $(4x^3 + 2y)f''(x^4 + y^2)$.
- A9.** In spherical coordinates, the volume element $dV \equiv dx dy dz$ is given by
(a) $\rho^2 \cos \phi \, d\rho d\phi d\theta$, (b) $\rho^2 \sin \theta \cos \phi \, d\rho d\phi d\theta$ (c) $\rho^2 \cot \phi \sin \theta \, d\rho d\phi d\theta$,
(d) $\rho^3 \cos \theta \, d\rho d\phi d\theta$, (e) $\rho^2 \sin \phi \, d\rho d\phi d\theta$.
- A10.** Let C be an arc whose parametric form is $x(t) = 1$, $y(t) = t$, $z(t) = t^2$ with $0 \leq t \leq 1$. Then the line integral $\int_C y \, dx + z \, dy + x^2y \, dz$ is equal to
(a) 0, (b) 1, (c) 2, (d) -1, (e) None of the above.
- A11.** Consider the function of period 2π defined on the interval $[-\pi, \pi]$ by

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \pi, \\ 0 & \text{if } -\pi < t < 0. \end{cases}$$

Then the Fourier expansion of f is

- (a) $\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nt}{2n}$, (b) $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nt}{n}$, (c) $\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{2n-1}$,
(d) $\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nt}{n}$, (e) None of the above.

A12. From the Fourier expansion $t \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nt$ with $-\pi < t < \pi$ and the identity

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12},$$

we have the following Fourier expansion for t^2 with $-\pi < t < \pi$:

(a) $t^2 \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t \sin nt,$ (b) $t^2 \sim \frac{\pi^2}{3} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos nt,$

(c) $t^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt,$ (d) $t^2 \sim 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt,$

(e) None of the above.

Part B: Do All Questions (6 Marks Each)

B1. Consider a curve defined by the parametric equations $x = e^t \cos t$, $y = e^t \sin t$, or, putting this in vector form, $\mathbf{r}(t) = e^t(\cos t \mathbf{i} + \sin t \mathbf{j}) \equiv e^t(\cos t, \sin t)$.

(a) Find the velocity vector $\mathbf{r}'(t)$, the acceleration vector $\mathbf{r}''(t)$ and the speed $v(t) \equiv |\mathbf{r}'(t)|$ at time t .

(b) Find the arc length of the portion of this curve between $t = 0$ and $t = 1$.

B2. Set up three equations for finding the closest point on the cubic curve $y^2 + y = x^3 + x^2 - 1$ to the origin $(0, 0)$ by using a Lagrange multiplier λ , and eliminate λ to reduce them to two equations. (The final answer for the closest point is not required; -do not try to find it.) Use the square of the distance between a point (x, y) on the curve and the origin as your objective function. Clearly indicate your constraint function.

B3. Locate all critical points of the function $f(x, y) = x^3y - xy + x^2$ and classify each of them (local maximum, local minimum or saddle point).

B4. Find the value of the double integral

$$\iint_D \frac{x^2}{1-y} dA,$$

where D is the region enclosed by the line $y = x$ and the cubic curve $y = x^3$, between $x = 0$ and $x = 1$.

B5. Use the polar coordinates to compute the moment of inertia

$$I_y = \iint_P y^2 dx dy,$$

where P is the semi-circular plate which is bounded by the y -axis and the semi-circle $x = \sqrt{2 - y^2}$.

B6. Show that the vector field $\mathbf{F} = (2 \sin x + y^2)\mathbf{i} + (2xy + \cos y)\mathbf{j}$ is conservative, (that is, \mathbf{F} is a gradient field.) Then find a potential function for this vector field.