

QUESTION 2. Find the volume of the solid obtained by rotating the function $f(x) = \sqrt{x}e^{-x^2}$ about the line $y = 0$ from $x = 1$ to $x = 2$.

- Interval of integration: $[1, 2]$.
- Radius and area of a cross section: a cross section $A(x)$, a disk in this case with radius $r(x) = f(x)$, has an area

$$A(x) = \pi r(x)^2 = \pi x e^{-2x^2}.$$

- Volume: an integration by substitution ($u = -2x^2$) gives the result

$$V = \int_1^2 A(x) dx = \frac{\pi}{4} (e^{-2} - e^{-8}).$$

QUESTION 3. Find the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2} dx.$$

- Domain: the domain of $\frac{1}{x^2+2x+2}$ is \mathbb{R} . This is an improper integral of type I.
- Antiderivative: first write $x^2 + 2x + 2 = (x + 1)^2 + 1$ and then substitute $u = x + 1 \Rightarrow du = dx$ to get

$$\int \frac{1}{x^2 + 2x + 2} = \int \frac{1}{u^2 + 1} = \arctan(u) + C = \arctan(x + 1) + C.$$

- Evaluation: we need first to decompose this integral. We have

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2} dx = \int_{-\infty}^0 \frac{1}{x^2 + 2x + 2} dx + \int_0^{\infty} \frac{1}{x^2 + 2x + 2} dx = I_1 + I_2$$

where

$$I_1 = \int_{-\infty}^0 \frac{1}{x^2 + 2x + 2} dx, \quad I_2 = \int_0^{\infty} \frac{1}{x^2 + 2x + 2} dx.$$

Now, we have by definition

$$I_1 = \lim_{L \rightarrow -\infty} \int_L^0 \frac{1}{x^2 + 2x + 2} dx, \quad \text{and} \quad I_2 = \lim_{U \rightarrow \infty} \int_0^U \frac{1}{x^2 + 2x + 2} dx$$

$$I_1 = \lim_{L \rightarrow -\infty} [\arctan(x + 1)]_L^0, \quad \text{and} \quad I_2 = \lim_{U \rightarrow \infty} [\arctan(x + 1)]_0^U$$

$$I_1 = \lim_{L \rightarrow -\infty} (-\arctan(1) - \arctan(L + 1)) = -\arctan(1) - \left(\frac{-\pi}{2}\right)$$

$$\text{and} \quad I_2 = \lim_{U \rightarrow \infty} (\arctan(1) + \arctan(U + 1)) = \arctan(1) + \frac{\pi}{2}.$$

- Conclusion: this improper integral converges to $-\arctan(1) + \pi/2 + \arctan(1) + \pi/2 = \pi$.

QUESTION 4. Calculate the following integral

$$\int_{-1}^2 \frac{1}{x^2 - x - 2} dx.$$

- The domain of $f(x) = \frac{1}{x^2 - x - 2} = \frac{1}{(x+1)(x-2)}$ is $D_f = \mathbb{R} \setminus \{-1, 2\}$. Hence, we have an improper integral between -1 and 2 .
- Antiderivative: we have to use partial fractions and solve for constantes A and B such that

$$f(x) = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1) + B(x-2)}{x^2 - x - 2} \Rightarrow A(x+1) + B(x-2) = 1$$

Now, from $A + B = 0$ and $A - 2B = 1$ we find $A = 1/3$ and $B = -1/3$. Therefore

$$f(x) = \frac{1}{3} \left(\frac{1}{x-2} - \frac{1}{x+1} \right)$$

and hence

$$\begin{aligned} \int f(x) dx &= \int \frac{1}{3} \left(\frac{1}{x-2} - \frac{1}{x+1} \right) dx = \frac{1}{3} \left(\int \frac{1}{x-2} dx - \int \frac{1}{x+1} dx \right) \\ &= \frac{1}{3} (\ln|x-2| - \ln|x+1|) + C = \frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| + C. \end{aligned}$$

- Decomposition: we have

$$\int_{-1}^2 f(x) dx = \int_{-1}^0 f(x) dx + \int_0^2 f(x) dx.$$

- Evaluation: we have by definition

$$\int_{-1}^0 f(x) dx = \lim_{l \rightarrow -1^+} \int_l^0 f(x) dx = \lim_{l \rightarrow -1^+} \left[\frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| \right]_l^0 = \infty.$$

- Conclusion: we stop here and conclude that this improper integral is divergent.

QUESTION 5. Calculate the following integral

$$\int_{-1}^1 \frac{1}{x^2} dx.$$

- The function $f(x) = \frac{1}{x^2}$ is not defined at zero. The domain is $D_f = \mathbb{R} \setminus \{0\}$, and so this integral is improper at 0.

- Antiderivative: we have

$$\int f(x) dx = \frac{-1}{x} + C.$$

- Decomposition: we have

$$\int_{-1}^1 f(x) dx = \int_{-1}^0 f(x) dx + \int_0^1 f(x) dx.$$

- Evaluation: by definition, we have

$$\int_{-1}^0 f(x) dx = \lim_{u \rightarrow 0^-} \int_{-1}^u f(x) dx = \lim_{u \rightarrow 0^-} \left(\frac{-1}{x} \right) \Big|_{-1}^u = \infty.$$

- Conclusion: we stop here and conclude that this improper integral is divergent.