MATH2004 D — Test 4: Nov 13, 10:35 - 11:25, Wed.

Name and Student Number:

Total points: 15. No partial marks for Questions 1-4.

Closed book! Non-programmer calculators are allowed!

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1. (1 point) Find the equation of the tangent plane of the surface $z = e^{x+y} + 2xy$ at the point $(1, -1, -1)$.

   (a) $x-3y-z=3$ (b) $x+3y+z=3$ (c) $x-3y+z=3$ (d) $x-2y+z=2$ (e) $x-2y+z=3$

   Solution: (c).
   $$z_x = e^{x+y} + 2y, \quad z_x(1,-1) = -1.$$
   $$z_y = e^{x+y} + 2x, \quad z_y(1,-1) = 3.$$
   Thus the equation of the tangent plane at the point $(1, -1, -1)$ is
   $$z - (-1) = (-1)(x-1) + (3)(y - -1), \quad i.e., \quad z = -x + 3y + 3.$$

2. (1 point) By changing the order of integration, the integral $\int_0^3 \int_{y^2}^9 f(x,y)dxdy =$

   (a) $\int_0^3 \int_0^{x^2} f(x,y)dydx$ (b) $\int_0^9 \int_0^{x^2} f(x,y)dydx$ (c) $\int_0^3 \int_0^{x^4} f(x,y)dydx$ (d) $\int_0^9 \int_0^{x^4} f(x,y)dydx$
   (e) $\int_0^9 \int_0^{\sqrt{x}} f(x,y)dydx$

   Solution: (e).
   The region is bounded by $x = 9$, $x = y^2$ and $y = 0$.

3. (1 point) Find the volume of the solid bounded by $z = \cos x \cos y$, the planes $x = \pi/4$, $y = \pi/4$, $x = 0$, $y = 0$, and $z = 0$.

   (a) $\frac{1}{2}$. (b) $\frac{1}{4}$. (c) $\frac{\sqrt{2}}{2}$. (d) 1. (e) 2.

   Solution: (a).
   $$V = \iint_R z dA = \int_0^{\pi/4} \int_0^{\pi/4} \cos x \cos y dx\, dy = 1/2.$$

4. (1 point) Find $\iint\limits_R \sqrt{x^2+y^2}\, dA$, where $R = \{(x,y) : x^2+y^2 \le 4,\ x \ge 0,\ y \ge 0\}$.

(a) $\sqrt{2}\pi$ (b) π (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$ (e) $\frac{4\pi}{3}$

Solution: (e).

$$\iint\limits_R \sqrt{x^2+y^2}\, dA = \int_0^{\pi/2}\int_0^2 rrdr\, d\theta = \frac{4\pi}{3}.$$

5. (4 points) Find $\iint\limits_R zdA$, where $z = x+2y$, R is the region bounded by $y = x+1$ and $y = x^2+1$.

Solution: The intersection points of $y = x+1$ and $y = x^2+1$: $x+1 = x^2+1, \Rightarrow x = 0, 1$. Thus $R = \{(x,y) : 0 \le x \le 1, x^2+1 \le y \le x+1\}$.

$$\iint\limits_R zdA = \int_0^1\int_{x^2+1}^{x+1}(x+2y)dy\, dx = \int_0^1 (-x^4 - x^3 + 2x)\, dx = -\frac{1}{5} - \frac{1}{4} + 1 = \frac{11}{20} = 0.55.$$

6. (7 points = 2+5) Let $f(x,y,z) = x+3y+2z$, $g(x,y,z) = 2x^2+2y^2+z^2$.

(i) Calculate $\bigtriangledown f(x,y,z)$ and $\bigtriangledown g(x,y,z)$.

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = 2x^2+2y^2+z^2 = 36$.

Solution: (i)

$$\bigtriangledown f(x,y,z) =< 1, 3, 2 >, \quad \bigtriangledown g(x,y,z) =< 4x, 4y, 2z > .$$

(ii) We solve

$$\bigtriangledown f - \lambda \bigtriangledown g = 0 \quad (\textbf{1 point for writing this equation.}) \tag{1}$$
$$g(x,y) = 36. \tag{2}$$

The equations (1)-(2) are equivalent to

$$1 - 4\lambda x = 0 \tag{3}$$
$$3 - 4\lambda y = 0 \tag{4}$$
$$2 - 2\lambda z = 0 \tag{5}$$
$$2x^2 + 2y^2 + z^2 = 36. \tag{6}$$

So we have

$$x = \frac{1}{4\lambda}, \qquad y = \frac{3}{4\lambda}, \qquad z = \frac{1}{\lambda}.$$

Hence

$$\frac{1}{8\lambda^2} + \frac{9}{8\lambda^2} + \frac{1}{\lambda^2} = 36,$$

which gives $\lambda = \pm\frac{1}{4}$.

When $\lambda = \frac{1}{4}$, we have $x = 1$, $y = 3$, and $z = 4$.

When $\lambda = -\frac{1}{4}$, we have $x = -1$, $y = -3$, and $z = -4$.

The maximum is $f(1, 3, 4) = 18$.

The minimum is $f(-1, -3, -4) = -18$.