Concordia University

Math 251 Class Test Solutions February 2006

Q1 12 marks Show that each of these is **not** a vector space:

a Under the operations inherited from $\mathbb{R}^3$, the set

$$S = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + y + z = 1 \}.$$

Solution: This is not a vector space. For example the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is in S, but $2\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ is not S since if we sum its entries we get $2 + 0 + 0 = 2$, and so S is not closed under scalar multiplication. (Note: there are other correct ways to answer this problem, for example we can also show that S is not closed under vector addition).

b Under the operations inherited from $\mathbb{R}^3$, the set

$$S = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0 \}.$$

Solution: This is not a vector space. For example the vectors $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are in S, but their sum $\mathbf{v}+\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is not in S since $1^2+1^2-2^2 = -2$ which is not zero. (Note: This set **is** closed under scalar multiplication and hence also under taking additive inverses).

Q2 12 marks

Let S be a subset of a vector space V. Prove that

$$Span(S) = \{c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + \ldots + c_n\mathbf{s}_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R},\ \ \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n \in S\}$$

is a subspace of V.

Solution: We proved a result that a subset of a vector space is a subset iff it is nonempty and closed under taking linear combinations of pairs of elements. Now for any subset S, $Span(S)$ (the notation is $[S]$ in the book) is nonempty: if S is the empty set then we defined its span to be $\{\mathbf{0}\}$, otherwise there is some element $\mathbf{s}$ in S and in this case the span contains $0.\mathbf{s} = \mathbf{0}$. So the span of S always contains the zero vector and so in particular is not the empty set.

Now suppose $\mathbf{v}$ and $\mathbf{w}$ are two elements of $Span(S)$ and a and b are two scalars. Then we have $\mathbf{v} = c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + \ldots + c_n\mathbf{s}_n$ for some scalars $c_1, c_2, \ldots, c_n \in I\!R$ and for some elements $\mathbf{s}_i \in S$, $i = 1 \ldots n$. Similarly $\mathbf{w} = c'_1\mathbf{s}'_1 + c'_2\mathbf{s}'_2 + \ldots + c'_m\mathbf{s}'_m$ for some scalars $c'_1, c'_2, \ldots, c'_m \in I\!R$ and for some elements $\mathbf{s}'_i \in S$, $i = 1 \ldots m$. So $a\mathbf{v} + b\mathbf{w} = ac_1\mathbf{s}_1 + ac_2\mathbf{s}_2 + \ldots + ac_n\mathbf{s}_n + bc'_1\mathbf{s}'_1 + bc'_2\mathbf{s}'_2 + \ldots + bc'_m\mathbf{s}'_m$. This is an element of $Span(S)$ and so S is closed under taking linear cominations of pairs of elements and so is a subspace of V.

Q3 12 marks

a For the set

$$S = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in I\!R^3 \mid 2x - y - z = 0\}$$

find two vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ such that $S = Span(\mathbf{s}_1, \mathbf{s}_2)$.

Solution: We have if

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$$

then $z = 2x - y$ and so

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ 2x - y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

So if we let $\mathbf{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{s}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ then we have show that S is a subset of $Span(\mathbf{s}_1, \mathbf{s}_2)$. Conversely every element of this span satisifies the condition to be in S and so S is equal to the span of $\mathbf{s}_1$ and $\mathbf{s}_2$.

b Show that $Span(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = I\!R^2$.

Solution: Every element of the span is a vector in $I\!R^2$ and so we have that $Span(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \subseteq I\!R^2$, and we also have to show that $I\!R^2 \subseteq$

$Span(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$. To show this we have to show that $\begin{pmatrix} x \\ y \end{pmatrix}$ is in $Span(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ for all x and y in $I\!R$. But

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x-y}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{x+y}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and so $I\!R^2 \subseteq Span(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ as required.

Q4 12 marks Show that the three functions $g_1(x) = 1$, $g_2(x) = 1 + x$ and $g_3(x) = x - x^2$ from $I\!R$ to $I\!R$ are linearly independent.

Solution: We proved a result which said that g_1, g_2 and g_2 are linearly independent iff the only solution to the equation $c_1g_1 + c_2g_2 + c_3g_3 =$ zero function is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. This is an equality of functions, so the function on the left and the function on the right must be the same when evaluated at every value of x. But the right side is the zero function and so always returns the value zero. Evaluating at, for example, $x = 0, 1, 2$ gives the equations $c1+c2 = 0$, $c1+2c_2 = 0$ and $c_1+3c_2-2c_3 = 0$. Solving we find that $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ and so the three functions are linearly independent. (Note: if you rewrite the left side as $(c_1 + c_2) + (c_2 + c_3)x - c_3x^2 =$ zero function and then claim that $c_1 + c_2 = 0$, $c_2 + c_3 = 0$ and $-c_3 = 0$, then you are assuming that $1, x, x^2$ are linearly independent functions (i.e. the only way of adding them to get zero is if the coefficients are zero). But this is essentially the same problem as the one we started with and you have to give some sort of argument to justify this.)

Q5 12 marks Define a new operation $\oplus$ on $I\!R$ by $x \oplus y = x + y - 1$ (so for example $2 \oplus 3 = 2 + 3 - 1 = 4$ and similarly for any other pair of numbers).

a Show that $x \oplus y = y \oplus x$ for all real numbers x and y.

Solution: For x and y in $I\!R$ we have $x \oplus y = x + y - 1 = y + x - 1 = y \oplus x$ using the commutativity of ordinary addition.

b Show that $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all real numbers x, y and z.

Solution: For all x, y and z in $I\!R$ we have $(x \oplus y) \oplus z = (x \oplus y) + z - 1 = x + y + z - 2 = x + (y \oplus z) - 1 = x \oplus (y \oplus z)$ using associativity and commutativity of ordinary addition.

c Show that there is exactly one number z such that $x \oplus z = x$ for all real numbers x.

Solution: For all $x \in I\!R$ we have $x \oplus 1 = x + 1 - 1 = x$ and so $z = 1$ has the required property. Moverover this is the only such number, since if z' is another number with this property then we have $z' = z' \oplus 1 = 1 \oplus z' = 1$.