Solution to Midterm Examination (version D)

MAT1330, Fall 2013

Question 1. [2 points] Find the value of a so that the following function is continuous

$$f(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1}, & x \neq 1 \\ a, & x = 1 \end{cases}$$

Justify your answer using limit laws.

Solution. $a = 4$. Since $\lim_{x\to 1}\dfrac{x^2+2x-3}{x-1} = \lim_{x\to 1}\dfrac{(x-1)(x+3)}{x-1} = \lim_{x\to 2}(x+3) = 4$.

Question 2. [8 points] A patient receives a daily dose $d = 5.4$mg of the drug FilGud R. In the course of 24 hours, 90% of the drug is absorbed and a fraction of $p = 0.1$ remains in the blood. The DTDS modelling the daily concentration M_t of FilGudR in the blood immediately after administering the dose is $M_{t+1} = pM_t + d = 0.1M_t + 5.4$.

(a) [0.5 point] The updating function of the DTDS is $f(x) = 0.1x + 5.4$.

(b) [1 point] The equilibrium point of the DTDS is $M^* = 6$, obtained by solving the equation $x = 0.1x + 5.4$, $0.9x = 5.4$, $x = 5.4 / 0.9 = 6$.

(c) [2 points] Give the general solution for the DTDS with general initial condition M_0:

$M_t = 0.1^t(M_0 - 6) + 6$.

This solution can be obtained in different ways.

(i) Use the formula $x_i = (x_0 - x^*)r^i + x^*$, as given in class. Now $x_0 = M_0$, $x^* = 6$ obtained in part (b), and $r = 0.1$.

(ii) Find the pattern of the solution using the first few terms.

(iii) Use the general solution $M_t = a(0.1)^t + b$.

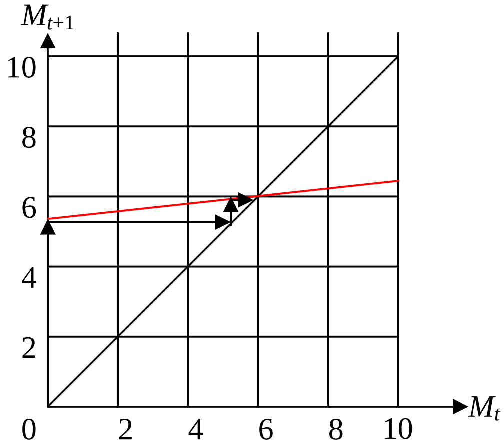
When $t = 0$, $M_0 = a + b$. When $t = 1$, $M_1 = 0.1a + b = 0.1M_0 + 5.4$. Hence, $0.1a + b = 0.1(a + b) + 5.4 = 0.1a + 0.1b + 5.4$. $0.9b = 5.4$. $b = 6$, and $a = M_0 - b = M_0 - 6$.

(d) [0.5 point] Calculate M_5 if $M_0 = 0$.

$M_1 = 0.1M_0 + 5.4 = 5.4$.
$M_2 = 0.1M_1 + 5.4 = 5.94$.
$M_3 = 0.1M_2 + 5.4 = 5.994$.

$M_4 = 0.1M_3 + 5.4 = 5.9994.$
$M_5 = 0.1M_4 + 5.4 = 5.99994.$

(e) [2 points] Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.



(f) [1 point] Is the equilibrium point stable or unstable?

Answer. Stable.

(g) [1 point] Due to sudden complications, the patient now also needs to take the drug WelSun R. This drug inhibits the uptake of FilGudR so that only 50% will be absorbed and a fraction of $p = 0.5$ will remain in the blood. Calculate the new daily dose d of FilGudR needed to maintain the equilibrium concentration of that drug at the same level as before.

Solution. The new system has updating function $f(x) = 0.5x + d$. The equilibrium point is at $x^* = 2d$. To maintain the same equilibrium as the original system, $2d = 6$. Hence, $d = 3$.

Question 3. [4 points] (a) Use the definition of the derivative to calculate the derivative of the function $f(x) = \dfrac{x}{6+2x}$.

Solution. $f'(x) = \lim_{h\to 0}\frac{1}{h}\left(\frac{x+h}{6+2(x+h)} - \frac{x}{6+2x}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{(x+h)(6+2x) - x(6+2(x+h))}{(6+2(x+h))(6+2x)}\right)$

$$= \lim_{h\to 0}\frac{6}{(6+2(x+h))(6+2x)} = \frac{6}{(6+2x)^2}.$$

(b) Check your answer by using differentiation rules.

Solution. Use the quotient rule.

$$f'(x) = \frac{(x)'(6+2x) - x(6+2x)'}{(6+2x)^2} = \frac{6+2x-2x}{(6+2x)^2} = \frac{6}{(6+2x)^2}.$$

Question 4. [6 points] Find the derivatives for the following functions. Do not simplify your answer.

(a) $f(x) = (3x^2 + 2)e^{-4x}$.

Solution. $f'(x) = 6xe^{-4x} + (3x^2 + 2)(-4)e^{-4x}$.

(b) $g(s) = \dfrac{\sin(7s+3)}{\cos(7s)}$.

Solution. $g'(s) = \dfrac{7\cos(7s+3)\cos(7s) - \sin(7s+3)(-7\sin(7s))}{\cos^2(7s)}$.

(c) $h(x) = \ln\left(\dfrac{y^{21}}{\sqrt{12y+2013}}\right)$.

Solution. Since $h(x) = 21 \ln y + \dfrac{1}{2}\ln(12y + 2013)$,

$h'(x) = \dfrac{21}{y} + \dfrac{12}{2(12y+2013)}$.

Question 5. [5 points] Consider the following function

$f(x) = \sqrt{5x}\, e^{-x/4}$.

(a) The domain of definition is: $[0, \infty)$.

(b) The critical point(s) is (are): $x = 0$, $x = 2$.

$$f'(x) = (\sqrt{5x})'e^{-x/4} + \sqrt{5x}\,(e^{-x/4})' = \frac{\sqrt{5}}{2\sqrt{x}}e^{-x/4} + \sqrt{5x}\left(-\frac{1}{4}\right)e^{-x/4} = e^{-x/4}\left(\frac{2\sqrt{5} - \sqrt{5}x}{4\sqrt{x}}\right).$$

Let $f'(x) = 0$, $x = 2$. When $x = 0$, $f'(x)$ does not exist.

(c) f is increasing in the interval: $(0, 2)$

$f'(x) > 0$ when $0 < x < 2$.

(d) f is decreasing in the interval: $(2, \infty)$

$f'(x) < 0$ when $x > 2$.

(e) Use a table of values to guess the horizontal asymptote.

x	4	9	25	49
$f(x)$	1.65	0.71	0.021	0.000075

The horizontal asymptote is $y = 0$.