

Material allowed: non programmable, non-graphing calculators.

Solve five problems of your choice

Self-serve formula sheet. For a vector-valued functions $\vec{r}(t)$ in three dimension defining a smooth curve for $t \in [a, b]$

$\kappa(t) =$	$a_T(t) =$	$s(t) =$	$\vec{T}(t) =$	$a_N(t) =$	$\vec{N}(t) =$	$\vec{B}(t) =$
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$\frac{\ \vec{r}'(t) \times \vec{r}''(t)\ }{\ \vec{r}'(t)\ }$ (a)	$\frac{\vec{r}'(t)}{\ \vec{r}'(t)\ }$ (b)	$\int_a^t \ \vec{r}'(\tau)\ d\tau$ (c)	$\frac{\ \vec{r}'(t) \times \vec{r}''(t)\ }{\ \vec{r}'(t)\ ^3}$ (d)	$\vec{T}(t) \times \vec{N}(t)$ (e)	$\frac{\vec{T}'(t)}{\ \vec{T}'(t)\ }$ (f)	$\frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\ \vec{r}'(t)\ }$ (g)
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[10 points] Problem 1.

(i) Find the velocity and the acceleration vectors $\vec{v}(t)$, $\vec{a}(t)$ for a particle that moves as described by the vector-valued function

$$\vec{r}(t) = t\mathbf{i} + [t \sin(t) + \cos(t)]\mathbf{j} + [t \cos(t) - \sin(t)]\mathbf{k}.$$

(ii) Compute the unit tangent $\vec{T}(t)$.

(iii) Compute the tangent component of the acceleration a_T .

[10 points] Problem 2.

Using the chain rule, compute the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ for the function f

$$f(u, v) = \cos(uv),$$

where the variables u, v depend on the variables x, y as follows

$$u = \ln(1 + x^2); \quad v = x^2 y.$$

[10 points] Problem 3.

For the given curve find the arclength parameter $s(t)$ and reparametrize the curve in terms of it

$$\vec{r}(t) = t^2\mathbf{i} + [t \sin(t) + \cos(t)]\mathbf{j} + [t \cos(t) - \sin(t)]\mathbf{k}, \quad t \geq 0.$$

[Advice: the ensuing integral is immediate if your derivatives are done properly]

[10 points] Problem 4.

(i) Compute the gradient of the function

$$F(x, y, z) = xy^2 \cos(yz).$$

(ii) Find a direction along which the function F increases as fast as possible at the point $(0, 1, \pi)$.

(iii) Compute the directional derivative at the point $(0, 1, \pi)$ along the direction of the vector $\vec{u} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$.

[10 points] Problem 5.

Find points on the surface $x^2 + 3y^2 + 4z^2 - 2xy = 16$ at which the tangent plane is parallel to the zx -plane.

[10 points] Problem 6.

Compute the divergence of the gradient of the following function of three variables

$$F(x, y, z) = x^2 yz - \frac{1}{3}y^3 z + z^2$$

Solution to Problem 1

We have

$$\begin{aligned}
\vec{v}(t) &= \mathbf{i} + t \cos(t)\mathbf{j} - t \sin(t)\mathbf{k} \\
\vec{a}(t) &= (\cos(t) - t \sin(t))\mathbf{j} + (-t \cos(t) - \sin(t))\mathbf{k} \\
\vec{T} = \frac{\vec{v}}{v} &= \frac{1}{\sqrt{1+t^2}} (\mathbf{i} + t \cos(t)\mathbf{j} - t \sin(t)\mathbf{k})
\end{aligned}$$

The tangent component is

$$a_T(t) = \vec{T} \cdot \vec{a} = \frac{t \cos^2(t) - t^2 \sin(t) \cos(t) + t \sin^2(t) + t^2 \sin(t) \cos(t)}{\sqrt{1+t^2}} = \frac{t}{\sqrt{1+t^2}}$$

Solution to Problem 2

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = \\
&= -v \sin(uv) \frac{2x}{1+x^2} - u \sin(uv) 2xy = -\sin(x^2 y \ln(1+x^2)) \left(\frac{2x^3 y}{1+x^2} + 2xy \ln(1+x^2) \right) \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = \\
&= -v \sin(uv) 0 - u \sin(uv) x^2 = -\sin(x^2 y \ln(1+x^2)) \ln(1+x^2) x^2
\end{aligned}$$

Solution to Problem 3

The arclength parameter is

$$s(t) = \int_0^t v(\tau) d\tau$$

Here the speed $v(\tau)$ is

$$\begin{aligned}
\vec{v}(\tau) &= 2\tau\mathbf{i} + \tau \cos(\tau)\mathbf{j} - \tau \sin(\tau)\mathbf{k} \\
v(\tau) &= \sqrt{4\tau^2 + \tau^2 \cos^2(\tau) + \tau^2 \sin^2(\tau)} = \sqrt{5\tau^2} = \tau\sqrt{5} \quad (\text{since } \tau \geq 0)
\end{aligned}$$

Therefore

$$s(t) = \int_0^t \tau\sqrt{5} d\tau = \frac{\sqrt{5}}{2} t^2$$

To reparametrize the curve we have to express t in terms of s . Solving the above relation for t we have

$$t(s) = \sqrt{\frac{2}{\sqrt{5}}} s$$

We now plug this into the expression for $\vec{r}(t)$

$$\frac{2}{\sqrt{5}} s \mathbf{i} + \left[\sqrt{\frac{2}{\sqrt{5}}} s \sin\left(\sqrt{\frac{2}{\sqrt{5}}} s\right) + \cos\left(\sqrt{\frac{2}{\sqrt{5}}} s\right) \right] \mathbf{j} + \left[\sqrt{\frac{2}{\sqrt{5}}} s \cos\left(\sqrt{\frac{2}{\sqrt{5}}} s\right) - \sin\left(\sqrt{\frac{2}{\sqrt{5}}} s\right) \right] \mathbf{k}$$

Solution to problem 4

We have

$$\vec{\nabla} F(x, y, z) = y^2 \cos(yz)\mathbf{i} + (2xy \cos(yz) - xy^2 z \sin(yz))\mathbf{j} - xy^3 \sin(yz)\mathbf{k}$$

At the point $(0, 1, \pi)$ the gradient is

$$\vec{\nabla} F(0, 1, \pi) = \cos(\pi)\mathbf{i} = -\mathbf{i}.$$

Since it is already normalized (length one) the direction of max increase at this point is $-\mathbf{i}$ (the gradient itself). To compute the directional derivative along $\vec{u} = \frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k})$ we have to compute

$$D_{\vec{u}} F|_{(0,1,\pi)} = \vec{\nabla} F(0, 1, \pi) \cdot \vec{u} = (-\mathbf{i}) \cdot \left(\frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k}) \right) = 0$$

Solution to Problem 5

For a plane to be parallel to the xz -plane it is necessary that the normal is -say- $\mathbf{j}$ (y -direction). That is the gradient of the function defining the surface must be parallel to $\mathbf{j}$ or equivalently must have zero components in the x and z direction. The gradient is

$$\vec{\nabla}F = (2x - 2y)\mathbf{i} + (6y - 2x)\mathbf{j} + 8z\mathbf{k}$$

Therefore we have to solve the system

$$\begin{cases} 2x - 2y = 0 \\ 8z = 0 \\ x^2 + 3y^2 + 4z^2 - 2xy = 16 \end{cases} \Rightarrow \begin{cases} x = y \\ z = 0 \\ 4x^2 - 2x^2 = 16 \Rightarrow x = \pm\sqrt{8} \end{cases}$$

There are thus two points as required, namely $(\sqrt{8}, \sqrt{8}, 0)$ and $(-\sqrt{8}, -\sqrt{8}, 0)$.

Solution to Problem 6

The gradient is

$$\vec{\nabla}F(x, y, z) = 2xyz\mathbf{i} - y^2z\mathbf{j} + \left(x^2y - \frac{1}{3}y^3 + 2z\right)\mathbf{k}$$

The divergence of the gradient is

$$\vec{\nabla} \cdot \vec{\nabla}F = \partial_x(\partial_x F) + \partial_y(\partial_y F) + \partial_z(\partial_z F) = 2yz - 2yz + 2 = 2.$$

Note that the end-result is a **scalar**