Solution to Midterm Examination 1 (version A)

MAT 1322D, Fall 2011

1. (3 marks) Use the definition of improper integrals to find the value of improper integral $\int_0^{\infty} \frac{x}{1+x^4}\,dx$.

Solution. (i) Let $u = x^2$, $u' = 2x$.

$$\int_0^{\infty} \frac{x}{1+x^4}\,dx = \lim_{b\to\infty}\int_0^{b} \frac{x}{1+x^4}\,dx = \lim_{b\to\infty}\frac{1}{2}\int_0^{b^2} \frac{1}{1+u^2}\,du = \frac{1}{2}\lim_{b\to\infty}\left[\arctan u\right]_{u=0}^{b^2} = \frac{1}{2}\lim_{b\to\infty}\arctan b^2 = \frac{\pi}{4}.$$

2. (3 marks) Use the comparison test to show that improper integral $\int_0^1 \frac{x+1}{x\sqrt{2x-x^2}}\,dx$ diverges.

Solution. When x is in interval (0, 1), $2x - x^2 < 2x$, and $x + 1 > 1$. $\frac{x+1}{x\sqrt{2x-x^2}} > \frac{1}{x\sqrt{2x}} = \frac{1}{\sqrt{2}x^{3/2}}$.

Since $\int_0^1 \frac{1}{x^{3/2}}\,dx$ diverges, improper integral $\int_0^1 \frac{x+1}{x\sqrt{2x-x^2}}\,dx$ diverges.

3. (3 marks) Let S be a solid with base bounded by the graph of $y = 3x - x^2$ and the x-axis in the interval [0, 3]. The cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

Solution. The area of a cross section at x is $A(x) = \frac{1}{2}\pi\left(\frac{3x-x^2}{2}\right)^2 = \frac{\pi}{8}(3x-x^2)^2$. Hence, the volume of the solid is

$$\frac{\pi}{8}\int_0^3 (9x^2-6x^3+x^4)dx = \frac{81\pi}{8}\left(1-\frac{3}{2}+\frac{3}{5}\right) = \frac{81\pi}{80}.$$

4. (3 marks) Recall that the length of the arc $y = f(x)$, $a \le x \le b$ is $L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$. Find the length of the arc $y = \frac{x^3}{12}+\frac{1}{x}$, $1 \le x \le 3$.

Solution. $y' = \frac{x^2}{4}-\frac{1}{x^2}$, $(y')^2 = \frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}$, $1+(y')^2 = \frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4} = \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2$.

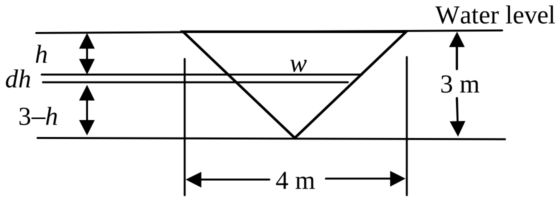
Hence, the length of the arc is $L = \int_1^3 \left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx = \left[\frac{x^3}{12}-\frac{1}{x}\right]_{x=1}^{3} = \frac{17}{6}$.

5. (4 marks) Find the work (in Joules) needed to build a stone column of diameter 2 m and height 10 m by lifting the stone from the ground level. (The density of stone is 4000 kg/m^3, and the acceleration of gravity g = 9.8 m/sec^2).

Solution. Consider a layer of stone in the column at a height H with thickness dH. The cross section is a circle with radius 1, whose area is $\pi(1)^2 = \pi$. The volume of this layer is $dV = \pi\, dH$. The weight of this layer is $dw = 4000\pi g\, dH$ Newtons. The work needed to lift the stone in this layer from the ground is $dW = 4000\pi g H dH$. Hence, the total work done is

$$W = 4000\pi g \int_0^{10} H dH = 4000\pi g \left[\frac{1}{2}H^2\right]_{H=0}^{10} \approx 6.16 \times 10^6 \text{ Joule.}$$

6. (4 marks) Find the force, in Newtons, acting on a triangular surface submerged into water as shown in the following figure:



(The density of water is 1000 kg/m^3, and g = 9.8).

Solution. Consider a slice with depth h of thickness dh. The width w of the slice is found by similar triangles: $\frac{w}{4} = \frac{3-h}{3}$, $w = \frac{4}{3}(3-h)$. The area is $dA = \frac{4}{3}(3-h)\, dh$. The force acting on this slice is $dF = \frac{4}{3}(3-h)\, \rho h dh$. The total force on this surface is

$$F = \frac{4000g}{3}\int_0^3 (3-h)h dh \approx 58800 \text{ Newtons.}$$