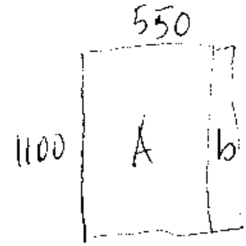
Mat 1341.F06 Final Solutions

1. Complete the following phrase to make a true statement:

"A system of 1100 linear equations in 550 unknowns..."



A. ... always has a solution. ×

B. ... always has a unique solution. ×

C. ... may be inconsistent. ✓

D. ... which is consistent always has a unique solution. ×

E. ... which is consistent never has a unique solution. ×

F. ... is never consistent. ×

2. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = 7$, find $\begin{vmatrix} 3a-5g & g & d \\ 3b-5h & h & e \\ 3c-5j & j & f \end{vmatrix} =$ $\begin{vmatrix} 3a & g & d \\ 3b & h & e \\ 3c & j & f \end{vmatrix} = 3 \begin{vmatrix} a & g & d \\ b & h & e \\ c & j & f \end{vmatrix}$

$= 3 \begin{vmatrix} a & b & c \\ g & h & j \\ d & e & f \end{vmatrix}$

A. 7

B. 21

C. -21

$= -3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix}$

D. 35

E. -35

$= -21$

3. If $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & 4 \end{bmatrix}$, then $(AB)^{-1} =$ $\quad B^{-1}A^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & 4 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$

A. $\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$ B. $\begin{bmatrix} 1/2 & -1/2 \\ -5 & 2 \end{bmatrix}$ C. $\begin{bmatrix} 1/2 & -5/2 \\ 2 & 3 \end{bmatrix}$ $\quad = \frac{1}{2}\begin{bmatrix} 1 & -5 \\ 4 & 6 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 \\ -5/2 & 3/2 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$ F. $\begin{bmatrix} -1 & -3/2 \\ 5/2 & 2 \end{bmatrix}$ $\quad = \begin{bmatrix} 1/2 & -5/2 \\ 2 & 3 \end{bmatrix}$

4. If c_j denotes the j^{th} column of

$$A = \begin{bmatrix} 1 & 4 & -1 & -2 \\ 3 & 1 & 0 & 5 \\ 4 & 5 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & -2 \\ 3 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & -2 \\ 0 & -11 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(j = 1, \ldots 4)$, which of the following sets is a basis for $U = \text{span}\{c_1, c_2, c_3, c_4\}$? $\quad \sim \begin{bmatrix} 1 & 4 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A. $\{c_1\}$
B. $\{c_2\}$
C. $\{c_3\}$
D. $\{c_1, c_2\}$
E. $\{c_1, c_2, c_3\}$
F. $\{c_1, c_3, c_4\}$

5. Compute $\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]^{2006}$

A

$A \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} = \begin{pmatrix} -I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = -I_4 \therefore A^2 = -I_4$

$\therefore A^{2006} = (-I_4)$

$= (-1)^{1003} I_4$

$= -I_4$

A. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ B. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ E. $\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ F. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

6. Let $\mathbf{M}_{22}$ denote, as usual,the vector space of real 2×2 matrices, and A^t denote the transpose of $A \in \mathbf{M}_{22}$. The dimension of $V = \{A \in \mathbf{M}_{22} \mid A = -A^t\}$ is:

A. 0

B. 1

C. 2

D. 3

E. 4

F. V is not a subspace of $\mathbf{M}_{22}$, so we cannot speak of its dimension.

$-\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\iff a = 0 \quad b = -c$

$d = 0$

$V = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$

$\dim V = 1$

7. The matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable over the reals. Why?

Evals are 1 & 2

$Ae_3 = 2e_3$

$[A - 2I] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (rank 2

$\therefore \dim \ker A - 2I = 1$

$\therefore \dim E_2 = 1$

$[A - I] \approx \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (rk = 2

$\therefore \dim E_1 = 1$

A. Because A does not have real eigenvalues. ×

B. Because A does not have any eigenvectors. × $Ae_3 = 2e_3$

C. Because A does not have three distinct eigenvalues.

D. Because A does not have three independent eigenvectors.

E. Because A is upper triangular. ×

F. Because there 'k' in 'ice-cream.' ×

$\dim E_1 + \dim E_2 = 2 < 3.$
$\therefore$ not enough l.i evecs.

8. The set of vectors $\{(0, 3, 4), (1, 0, 0), (0, 4, -3)\}$ is orthogonal. Find the Fourier coefficients (a_1, a_2, a_3) of $v = (0, -1, -1)$.

A. $(-\frac{7}{25}, 0, -\frac{1}{25})$

B. $(-\frac{7}{5}, 0, -\frac{1}{5})$

C. $(-7, 0, -1)$

D. $(0, -1, -1)$

E. $(\frac{7}{25}, 0, \frac{1}{25})$

F. $(-\frac{7}{25}, 0, \frac{1}{25})$

$a_1 = \dfrac{(0,3,4)\cdot(0,-1,-1)}{25} = -7/25$

$a_2 = \dfrac{(1,0,0)\cdot(0,-1,-1)}{1} = 0$

$a_3 = \dfrac{(0,4,-3)\cdot(0,-1,-1)}{25} = \dfrac{-1}{25}$

9. Let $W = \{(x, y, z, w) \in \mathbf{R}^4 \mid x\,y\,z\,w \leq 0\}$. Then,

A. $(0, 0, 0, 0) \in W$ but W is not closed under multiplication by scalars

B. $(0, 0, 0, 0) \notin W$ but W is closed under addition

C. W is closed under addition but W is not closed under multiplication by scalars

D. W is closed under addition and W is closed under multiplication by scalars

E. W is not closed under addition but W is closed under multiplication by scalars

F. None of the other statements is true.

Solution: (If you'd studied the W04 final, this was especially easy.)

A. is False: If $\mathbf{k} \in \mathbf{R}$ and $(x, y, z, w) \in W$ then $k(x, y, z, w) = (kx, ky, kz, kw) \in W$ because

$$kxkykzkw = k^4 xyzw \leq 0,$$

since $k^4 \geq 0$ and $xyzw \leq 0$.

So now <u>we know W is closed</u> <u>under multiplication by scalars.</u>

B. is False: since $(0, 0, 0, 0) \in W$.

C. is False: since we know already know W is closed under multiplication by scalars.

D. is False: $w = (-1, -1, 0, 0) \in W$ and $w' = (0, 0, 1, 1) \in W$ but $w + w' = (-1, -1, 1, 1) \notin W$.

So now <u>we know W is not closed under addition.</u>

E. is True: From what we learned from A and D.

So:

F. is False.

10. Suppose A is an $n \times n$ matrix. Among the following statements, which one is **not equivalent** to the others?

A. A is not invertible.

B. $Ax = 0$ has infinitely many solutions $x \in \mathbf{R}^n$.

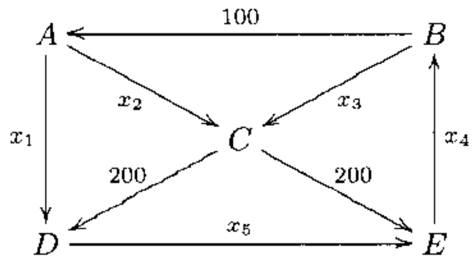
C. There is $b \in \mathbf{R}^n$ such that $Ax = b$ is inconsistent.

D. The determinant of A is zero.

E. A is row-equivalent to the identity matrix.

F. The rank of A is less than n.

11. Consider the network of streets and intersections below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave the intersections during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



3 a) Write down a system of linear <u>equations</u> which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)*

1 b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{cccccc} & & & & s & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & 0 & 1 & 300 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & 200 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints at this point.)

2 c) Find the maximum and minimum flow along BC, **using your results from (b).** *(You must justify all your answers.)*

a)

	Flow in	=	Flow out
A	100	=	$x_1 + x_2$
B	x_4	=	$100 + x_3$
C	$x_2 + x_3$	=	400
D	$x_1 + 200$	=	x_5
E	$200 + x_5$	=	x_4

2/5 each

; & $x_i \in \mathbb{Z}$ $i = 1, .., 5$ $\frac{1}{2}$

$x_i \geq 0$ " . $\frac{1}{2}$

b) $x_1 = -200 + s$
$x_2 = 300 - s$
$x_3 = 100 + s$; $s \in \mathbb{R}$
$x_4 = 200 + s$
$x_5 = s$

1/5 each

c) Flow along BC is $x_3 = 100 + s$. Now $x_1 \geq 0 \Leftrightarrow s \geq 200$ while $x_2 \geq 0$ $\Leftrightarrow s \leq 300$. No other constraints change this. Hence $200 \leq s \leq 300$

$\therefore 300 \leq 100 + s \leq 400$

1 - Correct answers
1 - justification

Max flow along BC $= 400$
Min $= 300$.

12. Let $U = \{(x,y,z,w) \in \mathbf{R}^4 \mid x + z + w = 0\}$. $[1\ 0\ 1\ 1 | 0]$

2 a) Find a basis of U and give the dimension of U.

2 b) Find an orthogonal basis of U.

2 c) Find the best approximation to $(1,1,0,0)$ by vectors in U.

$1\frac{1}{2}$ - justified basis   1 - no just

a) $x = -s - t$, $y = r$, $z = s$, $w = t$; $r, s, t \in \mathbb{R}$; $\therefore \{\overset{v_1}{(-1,0,1,0)}, \overset{v_2}{(-1,0,0,1)}, \overset{v_3}{(0,1,0,0)}\}$ is a basis of U.

$\therefore \underline{\dim U = 3}$ $\frac{1}{2}$ (consistent with ↑)

b) $w_1 = v_1$

$w_2 = v_2 - \frac{w_1 \cdot v_2}{\|w_1\|^2} w_1 = (-1,0,0,1) - \frac{1}{2}(-1,0,1,0)$ $\frac{1}{2}$: G.S

$= (-\tfrac{1}{2}, 0, -\tfrac{1}{2}, 1)$

$w_3 = v_3 - \overset{0}{\frac{\cancel{w_1 \cdot v_3}}{\|w_1\|^2}} w_1 - \overset{0}{\frac{\cancel{w_2 \cdot v_3}}{\|w_2\|^2}} w_2 = v_3 = (0,1,0,0)$

$\therefore \{\overset{w_1}{(-1,0,1,0)}, \overset{\tilde{w}_2}{(-1,0,-1,2)}, \overset{w_3}{(0,1,0,0)}\}$ is an orthogonal basis of U. $\frac{1}{2}$, 1 orthog $\frac{1}{2}$ $\subset U$

c) $\underbrace{\mathrm{proj}_U}_{1}(1,1,0,0) = \frac{(1,1,0,0)\cdot w_1}{\|w_1\|^2} w_1 + \frac{(1,1,0,0)\cdot \tilde{w}_2}{\|\tilde{w}_2\|^2} \tilde{w}_2 + \frac{(1,1,0,0)\cdot w_3}{\|w_3\|^2} w_3$

$= \frac{-1}{2} w_1 + \frac{-1}{6}\tilde{w}_2 + 1 \cdot w_3$

$= (\tfrac{1}{2}, 0, -\tfrac{1}{2}, 0) + (\tfrac{1}{6}, 0, \tfrac{1}{6}, -\tfrac{1}{3}) + (0,1,0,0)$ = $(\tfrac{2}{3}, 1, -\tfrac{1}{3}, -\tfrac{1}{3})$

$= \tfrac{1}{3}(2,3,-1,-1)$

1 - correct, $\frac{1}{2}$ $\in U$

13. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

1 a) Find the characteristic polynomial $\det(A - \lambda I)$ of A, and deduce that the eigenvalues of A are 1 and 2.

1 b) Find a basis of $E_1 = \{x \in \mathbf{R}^3 \mid Ax = x\}$.

2 c) Find a basis of $E_2 = \{x \in \mathbf{R}^3 \mid Ax = 2x\}$.

2 d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Explain why your choice of P is invertible.

a) $|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & \lambda-2 \\ 1 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)^2 \begin{vmatrix} -\lambda & -1 \\ 1 & 1 \end{vmatrix}$

$= \underline{(2-\lambda)^2(-\lambda+1)}$ (1)

($\frac{1}{2}$ if merely check 1&2 give 0 here)

$\therefore |A - \lambda I| = 0 \Leftrightarrow \lambda = 2$ or $\lambda = 1$

b) $E_1 = \ker(A - I) = \ker\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

on $= \{(-2s, s, s) \mid s \in \mathbb{R}\}$ ($\frac{1}{2}$) $\therefore \{(-2,1,1)\}$ ($\frac{1}{2}$)

is a basis for E_1

c) $E_2 = \ker(A - 2I) = \ker\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \ker \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = \{(-t, s, t) \mid s, t \in \mathbb{R}\}$ $\frac{1}{2}$

$\frac{1}{2}$ - rank 1

$\therefore \{(-1,0,1),(0,1,0)\}$ 1 is a basis for E_2

d) Let $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (1), $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ($\frac{1}{2}$). Then $P^{-1}AP = D$.

$\det P = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -(-1) = 1 \neq 0 \therefore P$ is invertible

($\frac{1}{2}$)

14. Define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y \\ y+z \\ z-x \end{bmatrix}.$$

$1\frac{1}{2}$ a) Find the standard matrix of S.

$1\frac{1}{2}$ b) Find a basis for $\ker S$.

$1\frac{1}{2}$ c) Give a complete geometric description of $\ker S$.

$1\frac{1}{2}$ d) Find $\dim \operatorname{im} S$.

$1\frac{1}{2}$ a) $A = (Se_1\ Se_2\ Se_3) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ $1\frac{1}{2}$

$A^t = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$1\frac{1}{2}$ b) $\ker S = \ker A = \ker \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\frac{1}{2}$

$\ker A^t = \ker A$!

$= \left\{ \begin{bmatrix} s \\ -s \\ s \end{bmatrix} \middle| s \in \mathbb{R} \right\}$ $\therefore$ (1) $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a basis for $\ker S$

$1\frac{1}{2}$ c) $\ker S$ is the line through 0 in $\mathbb{R}^3$ with dir $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$1\frac{1}{2}$ d) $\dim \operatorname{im} S = \underline{\dim \operatorname{col} A = \operatorname{rank} A} = \underbrace{2}_{(1)}$.

$\left(\frac{1}{2}\right)$ justn

Lou wi a)

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

15. State whether the following are true or false. If true, explain why, if false, give an explicit counter-example to illustrate.

(a) If u, v and w are three linearly independent vectors in a vector space E, then $u \in \operatorname{span}\{u - v, v - w\}$.

$\boxed{\text{FALSE}}$

$$u = a(u - v) + b(v - w)$$

$$\Leftrightarrow \quad 0 = (a-1)u + (-a+b)v - bw$$

$$\Leftrightarrow \quad \left.\begin{array}{l} a - 1 = 0 \\ -a + b = 0 \\ -b = 0 \end{array}\right\} \quad a = 1 = b = 0$$

which is impossible

(b) Let A be a real 3×2 matrix and suppose there is a vector $v \in \mathbf{R}^2$ with $v \neq 0$ and $Av = 0$. Then the columns of A are dependent.

$\boxed{\text{TRUE}}$

$$A = [c_1 \; c_2] \qquad A\begin{bmatrix} x \\ y \end{bmatrix} = xc_1 + yc_2 = 0$$

with $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \{c_1, c_2\}$ dependent.

c) Suppose B is an invertible 3×3 matrix, and v_1, v_2, and v_3 span $\mathbf{R}^3$. Then $\{Bv_1, Bv_2, Bv_3\}$ also spans $\mathbf{R}^3$.

$\boxed{\text{TRUE}}$

$\{v_1, v_2, v_3\}$ span $\mathbb{R}^3 \iff A = [v_1\ v_2\ v_3]$ is invertible.

But $[Bv_1\ Bv_2\ Bv_3] = B[v_1\ v_2\ v_3] = BA$ is therefore invertible too, so $\{Bv_1, Bv_2, Bv_3\}$ spans $\mathbb{R}^3$

d) Let A be a 4×6 matrix, and define a linear transformation $T_A : \mathbf{R}^6 \to \mathbf{R}^4$ by $T_A(v) = Av$, for all $v \in \mathbf{R}^6$. Then $\ker T_A \neq \{0\}$.

$\boxed{\text{TRUE}}$

$\dim \ker T_A + \dim \operatorname{im} T_A = 6$

But $\operatorname{im} T_A$ is a s.s. of $\mathbb{R}^4$ $\therefore \dim \operatorname{im} T_A \leq 4$

$\therefore \dim \ker T_A \geq 2$

$\therefore \quad \ker T_A \neq 0$

16. (4 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.

Let A be a symmetric $n \times n$ matrix.

a) Prove that $(Au) \cdot u' = u \cdot (Au')$ for all $u, u' \in \mathbf{R}^n$. (Here "$\cdot$" denotes the dot product)

Let u, u' be written as column vectors. Then

2

$$(Au) \cdot u' = \underbrace{(Au)^t u'}_{\text{matrix product}} = u^t A^t u' = u^t A u' = \underbrace{u \cdot (Au')}_{\text{dot product}}$$

$$\therefore (Au) \cdot u' = u \cdot (Au')$$

Now let $v_\lambda \in \mathbf{R}^n$ be an eigenvector of A with eigenvalue λ, and set $W = \{w \in \mathbf{R}^n \mid w \cdot v_\lambda = 0\}$.

b) Prove that if $w \in W$, then $Aw \in W$.

2

Consider $(Aw) \cdot v_\lambda$. By (a) $(Aw) \cdot v_\lambda$

$= w \cdot (Av_\lambda)$

$= w \cdot (\lambda v_\lambda)$

$= \lambda (w \cdot v_\lambda)$

$= \lambda (0)$ (since $w \in W$)

$= 0$

$\therefore (Aw) \cdot v_\lambda = 0$, so $Aw \in W$.