Université d'Ottawa
Faculté de Génie

Département de
Génie Mécanique

University of Ottawa
Faculty of Engineering

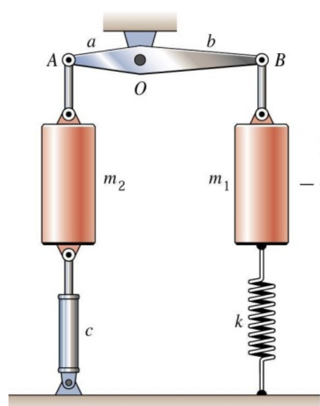
Department of
Mechanical Engineering

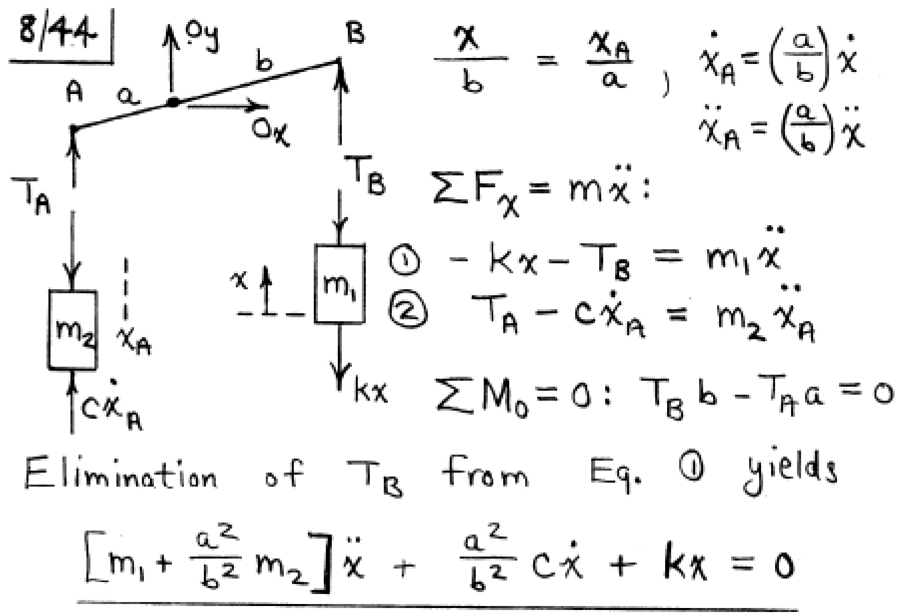
MCG 4308
Mechanical Vibration Analysis

MIDTERM EXAMINATION
February 14, 2013
SOLUTIONS

Question 1

a) Derive the differential equation of motion in terms of the variable *x* for the system shown in the figure. Neglect the mass of the link AB and assume small oscillations.
b) What is the equivalent mass of the system?
c) What is the damping ratio for the system?
d) What is the natural frequency of the system?
e) Will the system shown in the figure oscillate **at** the natural frequency or at another frequency? Explain briefly.





Equivalent mass is given by $m_1 + \dfrac{a^2}{b^2} m_2$

Damping ratio is given by $\zeta = \dfrac{c_{eq}}{2\sqrt{km_{eq}}} = \dfrac{a^2 c}{b^2 2\sqrt{k\left(m_1 + \dfrac{a^2}{b^2} m_2\right)}} = \dfrac{a^2 c}{2b\sqrt{k\left(b^2 m_1 + a^2 m_2\right)}}$

Natural frequency of the system is given by $\sqrt{\dfrac{k}{m_{eq}}} = \sqrt{\dfrac{k}{m_1 + \dfrac{a^2}{b^2} m_2}} = b\sqrt{\dfrac{k}{b^2 m_1 + a^2 m_2}}$

The system here is damped so will not oscillate at the natural frequency. If the system is underdamped, it will oscillate at the damped frequency given by $\omega_d = \omega_n\sqrt{1-\zeta^2}$. If it is critically damped or overdamped, there will be no oscillators but an exponential decay in the displacement.

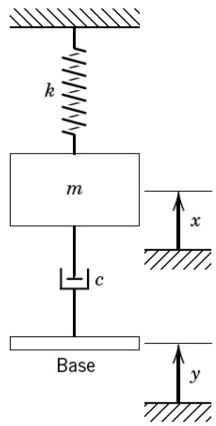
Question 2

The figure shows a system being driven by base excitation through a damping element. Assume that the base displacement is sinusoidal: $y(t)=Y\sin(\omega t)$.

a) Derive the equation of motion for this system.
b) Derive the expression for X, the steady-state amplitude of the motion of the mass m. Express your answer in terms of the damping ratio ζ and the frequency ratio $r = \omega/\omega_n$ where ω_n is the natural frequency of the system.

c) Derive the expression for F_t, the steady-state amplitude of the force transmitted to the fixed support (the wall at top). Express your answer in terms of ζ and $r = \omega/\omega_n$, as for (b).

d) Find the Displacement Transmissibility and the Force Transmissibility for this system.



Problem 4.17 a) From Newton's law,

$$m\ddot{x} = c(\dot{y} - \dot{x}) - kx$$

or

$$m\dot{x} + c\dot{x} + kx = c\dot{y}$$

$$T(s) = \frac{X(s)}{Y(s)} = \frac{cs}{ms^2 + cs + k}$$

Thus

$$T(i\omega) = \frac{c\omega i}{-m\omega^2 + c\omega i + k} = \frac{c\omega i/k}{1 - r^2 + c\omega i/k}$$

This gives

$$T(i\omega) = \frac{\frac{c\omega m}{mk} i}{1 - r^2 + c\omega mi/mk} = \frac{2\zeta\omega_n \omega i/\omega_n^2}{1 - r^2 + 2\zeta\omega_n\omega i/\omega_n^2} = \frac{2\zeta r i}{1 - r^2 + 2\zeta r i}$$

where we have used the fact that $\omega_n = \sqrt{k/m}$, $r = \omega/\omega_n$, and $c/m = 2\zeta\omega_n$. The magnitude is

$$X = \frac{2\zeta r}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} Y$$

b) Thus

$$F_t = kX = \frac{2\zeta r k}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} Y$$

Displacement transmissibility is given by $\frac{X}{Y} = \frac{2\zeta r}{\sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}}$

The force transmissibility is given by $\frac{F_t}{Y} = \frac{2\zeta r k}{\sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}}$

Question 3

In an effort to understand weird observed behaviour in a mechanical component, the design team resorts to modelling the component using a simple 1 DoF model that has *negative* damping. Their modelling efforts produce the following equation of motion:

$$\ddot{x} - \dot{x} + x = 0$$

a) Find the response of the system assuming initial conditions of $x_0 = 1$ and $v_0 = 0$.
b) *Sketch* the response of the system you obtained in part (a).
c) The team concludes that the vibrations of the mechanical component are undesirably large, which may cause damage. Which of the following techniques can be used, with confidence, to reduce the motion of the component? Choose ALL that apply and explain briefly your reasoning.
   A. Reduce the mass of the component
   B. Increase the mass of the component
   C. Reduce the stiffness of the component
   D. Increase the stiffness of the component
   E. Add an external damper to the component

Solution: This is a problem with negative damping which can be used to tie into Section 1.8 on stability, or can be used to practice the method for deriving the solution using the method suggested following equation (1.13) and eluded to at the start of the section on damping. To this end let $x(t) = Ae^{\lambda t}$ the equation of motion to get:

$$(\lambda^2 - \lambda + 1)e^{\lambda t} = 0$$

This yields the characteristic equation:

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j, \text{ where } j = \sqrt{-1}$$

There are thus two solutions as expected and these combine to form

$$x(t) = e^{0.5t}(Ae^{\frac{\sqrt{3}}{2}jt} + Be^{-\frac{\sqrt{3}}{2}jt})$$

Using the Euler relationship for the term in parenthesis as given in Window 1.4, this can be written as

$$x(t) = e^{0.5t}(A_1 \cos\frac{\sqrt{3}}{2}t + A_2 \sin\frac{\sqrt{3}}{2}t)$$

Next apply the initial conditions to determine the two constants of integration:

$$x(0) = 1 = A_1(1) + A_2(0) \Rightarrow A_1 = 1$$

Differentiate the solution to get the velocity and then apply the initial velocity condition to get

$\dot{x}(t) =$

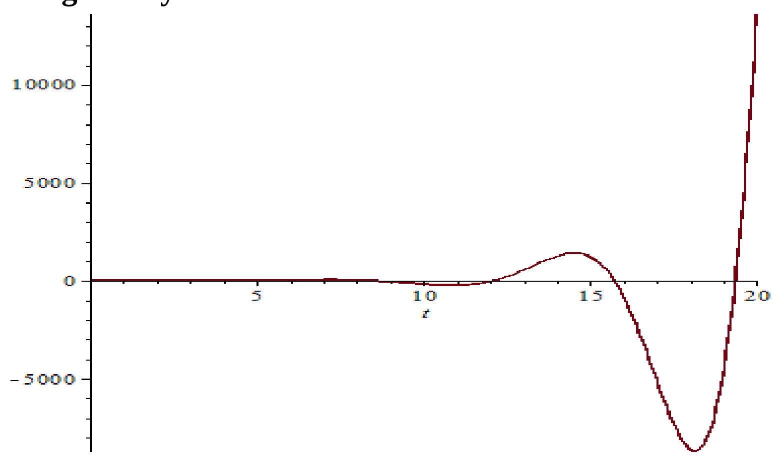
$$\frac{1}{2}e^{0}(A_1\cos\frac{\sqrt{3}}{2}0 + A_2\sin\frac{\sqrt{3}}{2}0) + e^{0}\frac{\sqrt{3}}{2}(-A_1\sin\frac{\sqrt{3}}{2}0 + A_2\cos\frac{\sqrt{3}}{2}0) = 0$$

$$\Rightarrow A_1 + \sqrt{3}(A_2) = 0 \Rightarrow A_2 = -\frac{1}{\sqrt{3}},$$

$$\Rightarrow x(t) = e^{0.5t}(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)$$

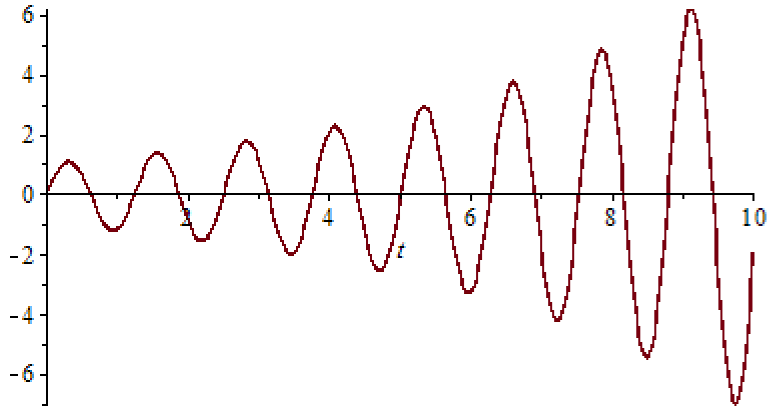
This function oscillates with increasing amplitude as shown in the following plot which shows the increasing amplitude. This type of response is referred to as a flutter instability. This plot is from Mathcad.

The plot from MAPLE is given by



In this particular problem, the solution blows up much more rapidly than it is oscillating, so you won't see a lot of "wiggles" within the expnential blowing up.

However, a typical "sketch" for this kind of solution (exponential GROWTH and sinusoidal oscillations) would be



From an energy point of view, (positive, normal) damping removes energy from the system so must always be stable. Thus, using the same point of view, positive damping can be considered to be something that always ADDS energy to the system. From that point of view, changing the mass or stiffness of the system will not solve the problem because energy is always being added. Hence, the only possivle solutions is (E) with the goal to add sufficient external damping to the system to yield a NET positive damping term in the dynamics of the system (which can be seen from the equation of motion). At worst, we can add sufficient damping to the system so that the "damping" term (coefficient of $\dot{x}$) is effectively zero which will yield a regular oscillating system.

Potentially Useful formulae

Quadratic formula: roots of $ax^2 + bx + c$ are given by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\sin(u) = \frac{e^{iu} - e^{-iu}}{2i} = \mathrm{Im}\left\{e^{iu}\right\} \qquad \cos(u) = \frac{e^{iu} + e^{-iu}}{2} = \mathrm{Re}\left\{e^{iu}\right\}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\,\omega_f t} \quad \text{with} \quad c_n = \frac{1}{T}\int_0^T f(t) e^{-in\omega_f t} dt \qquad \omega_f = 2\pi/T$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_f t) + b_n \sin(n\omega_f t)\right]$$

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{n2\pi t}{T}\right)dt \qquad b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{n2\pi t}{T}\right)dt$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \text{ for all } x$$