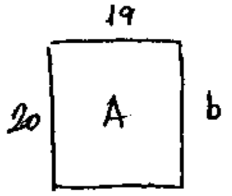
F05 1341A Final: Solutions

1. For a non-homogeneous system of 20 equations in 19 unknowns, answer the following three questions:

- ○ Can the system be inconsistent? YES
- ○ Can the system have infinitely many solutions? YES
- ○ Can the system have only one solution? YES



A. Yes, Yes, No.

B. No, No, Yes.

C. Yes, No, Yes.

D. No, Yes, Yes.

[✓ E.] Yes, Yes, Yes.

F. No, No, No.

$0 \le \operatorname{rank} A \le 19$ ∴ $Ax=b$ can have a unique sol^n, or infinitely many. It may also have no sol^n if $\operatorname{rank} A < \operatorname{rank}[A|b]$

2. If $\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 13$, find $\begin{vmatrix} 2a-3g & g & d\\ 2b-3h & h & e\\ 2c-3i & i & f\end{vmatrix}$.

$$\overset{3C_2+C_1\to C_1}{=} \begin{vmatrix} 2a & g & d\\ 2b & h & e\\ 2c & i & f\end{vmatrix} = 2\begin{vmatrix} a & g & d\\ b & h & e\\ c & i & f\end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c\\ g & h & i\\ d & e & f\end{vmatrix} = -2\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}$$

$$= -2(13) = -26$$

A. 13

B. 26

[✓ C.] -26

D. 39

E. -39

F. -13

3. Let $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then the second row of B^{-1} is:

[✓ A.] [0 1 -1]

B. [-1 1 0]

C. [0 -1 1]

D. [1 -1 0]

E. [1 0 -1]

F. None of the above

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & \overset{*}{1} & 0 & 0 & \overset{*}{1} & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

4. If $v_1 = (1,3,4), v_2 = (2,6,8), v_3 = (4,1,5)$ and $v_4 = (-2,5,3)$, which of the following is a basis for $U = \text{span}\{v_1, v_2, v_3, v_4\}$?

A. $\{v_1\}$

[✓ B.] $\{v_1, v_3\}$

C. $\{v_2, v_3, v_4\}$

D. $\{v_1, v_2, v_3\}$

E. $\{v_1, v_3, v_4\}$

F. $\{v_1, v_2, v_3, v_4\}$

$$\text{Let } A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 4 & -2 \\ 3 & 6 & 1 & 5 \\ 4 & 8 & 5 & 3 \end{bmatrix} \\ \begin{array}{cccc} v_1 & & v_3 & \end{array} \end{array} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 0 & -11 & 11 \\ 0 & 0 & -11 & 11 \end{bmatrix}$$

$$\sim \begin{bmatrix} \textcircled{1} & 2 & 4 & -2 \\ 0 & 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \{v_1, v_3\}$ is a basis for U, by the column-space algorithm.

5. Let $A = \begin{bmatrix} a & -2 \\ 2 & b \end{bmatrix}$. Find all (a, b) for which $A^2 = 0$.

A. $\pm(1, 1)$
[✓ B.] $\pm(2, -2)$
C. $\pm(2, 2)$
D. $\pm(3, -3)$
E. $\pm(3, 3)$
F. There is no such (a,b).

$$A^2 = \begin{bmatrix} a & -2 \\ 2 & b \end{bmatrix}\begin{bmatrix} a & -2 \\ 2 & b \end{bmatrix} = \begin{bmatrix} a^2-4 & -2a-2b \\ 2a+2b & b^2-4 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

∴ need $a^2 = 4$

$a = -b$ ∴ $a = 2$, $b = -2$

$b^2 = 4$ or $a = -2$, $b = 2$

6. Let $\mathbf{M}_{22}$ denote, as usual,the vector space of real 2×2 matrices and let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The dimension of $C = \{A \in \mathbf{M}_{22} \mid JA = AJ\}$ is:

A. 0
B. 1
[✓ C.] 2
D. 3
E. 4
F. 5

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$$

∴ $b = -c$
~~$-a = -d$~~
$a = d$
~~$b = -c$~~

ie. $C = \left\{ \begin{bmatrix} d & -c \\ c & d \end{bmatrix} \middle| \ c, d \in \mathbb{R} \right\}$

$= \text{Span}\left\{ \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{J}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \right\}$

Moreover $\{I, J\}$ is l.i., so $\dim C = 2$

7. The matrix $A = \begin{bmatrix} 8 & -3 \\ 18 & -7 \end{bmatrix}$ is diagonalizable. Which of the following could be $P^{-1}AP$ for some invertible matrix P?

A. $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ [✓ B. $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$] C. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ E. $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ F. $\begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix}$

$|A-\lambda I| = \begin{vmatrix} 8-\lambda & -3 \\ 18 & -7-\lambda \end{vmatrix} = \lambda^2 - \lambda - 56 + 54 = \lambda^2 - \lambda - 2$

$= (\lambda+1)(\lambda-2)$

∴ $\lambda = -1$, $\lambda = 2$
are the evals of A

∴ if $P^{-1}AP$ is diagonal,
it could only be
$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, of the choices above

8. Which of the following are subspaces of $\mathbf{M}_{22}$?

(1) The set of all symmetric 2×2 matrices (i.e. when $A = A^t$). ✓ (1) = $\left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle| a, b, d \in \mathbb{R} \right\}$

(2) The set of all anti-symmetric 2×2 matrices (i.e. when $A = -A^t$) ✓ (2) = $\left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

(3) The set of all invertible 2×2 matrices ✗ see below

(4) The set of all 2×2 matrices with trace 0. (Recall that the trace of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $a + d$.) ✓
(seen in class)

A. (1) and (2)
B. (1) and (3)
C. (1), (2) and (3)
D. (2), (3) and (4)
E. (3) and (4)
[✓ F.] (1), (2) and (4)

(1) = span$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ & ∴ is a s.s. of M_{22}

(2) = Span$\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$, so is a ss of M_{22}

(3) : I_2 is inv and $-I_2$ is inv.
but $0 = I_2 + (-I_2)$ is
not ∴ (3) is not closed
under add^n; OR I_2 is inv. but $0.I_2 = 0$
is not ∴ (3) is not a s.s. of M_{22}

(4) We saw this in class (it was called Sl_2)

$Sl_2 = \left\{ \begin{bmatrix} -d & b \\ c & d \end{bmatrix} \middle| b, c, d \in \mathbb{R} \right\} = $ span$\left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$,

& so is a s.s. of M_{22}

9. Let $\mathbf{F}[\mathbf{R}] = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$ be the vector space of all real-valued funtions of a real variable. It is known that $\{\sin x, \cos x\}$ is linearly independent in $\mathbf{F}[\mathbf{R}]$, and that $\sin(x+a) = \cos x \sin a + \sin x \cos a$, for all $x, a \in \mathbf{R}$.

*

What are the dimensions of

$$V = \text{span}\{\sin x, \cos x, \sin(x+1)\} \quad \text{and} \quad W = \text{span}\{2\sin x, 3\cos x\}?$$

A. $\dim V = 3, \dim W = 3$

B. $\dim V = 2, \dim W = 3$

C. $\dim V = 3, \dim W = 2$

[✓ D.] $\dim V = 2, \dim W = 2$

E. $\dim V = 1, \dim W = 3$

F. $\dim V = 1, \dim W = 2$

By the formula *

$\sin(x+1) = \sin(1)\cos x + \cos(1)\sin x \in \text{span}\{\sin x, \cos x\}$

Hence $V = \text{span}\{\sin x, \cos x\}$. Since $\{\sin x, \cos x\}$ is l.i, $\dim V = 2$.

Moreover, $W = \text{span}\{2\sin x, 3\cos x\} = \text{span}\{\sin x, \cos x\} = V$

$\therefore \dim W = \dim V = 2$

10. Suppose A is an $n \times n$ matrix. Among the following statements, which one is **not equivalent** to the others?

A. A is invertible.

[✓ B.] $Ax = 0$ has a non-trivial solution $x \in \mathbf{R}^n$. is the statement not equivalent to the others

C. $Ax = b$ has a unique solution $x \in \mathbf{R}^n$ for every $b \in \mathbf{R}^n$.

D. The determinant of A is not zero.

E. A is row-equivalent to the identity matrix.

F. The rank of A is n.

11. (a) Consider the linear system

$$\begin{array}{rcrcrcr} x & & & + & z & = & -1 \\ 2x & - & y & & & = & 2 \\ & & y & + & 2z & = & -4 \\ ax & + & cy & + & az & = & 0 \end{array}$$

Find all a and c so that this system has

(i) a unique solution,
(ii) infinitely many solutions, and
(iii) no solutions.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ a & c & a & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & -2 & 4 \\ 0 & 1 & 2 & -4 \\ 0 & c & 0 & a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & c & 0 & a \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & -2c & a+4c \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \therefore \text{rank} A = \begin{cases} 2 & \text{if } c = 0 \\ 3 & \text{if } c \neq 0 \end{cases}$$

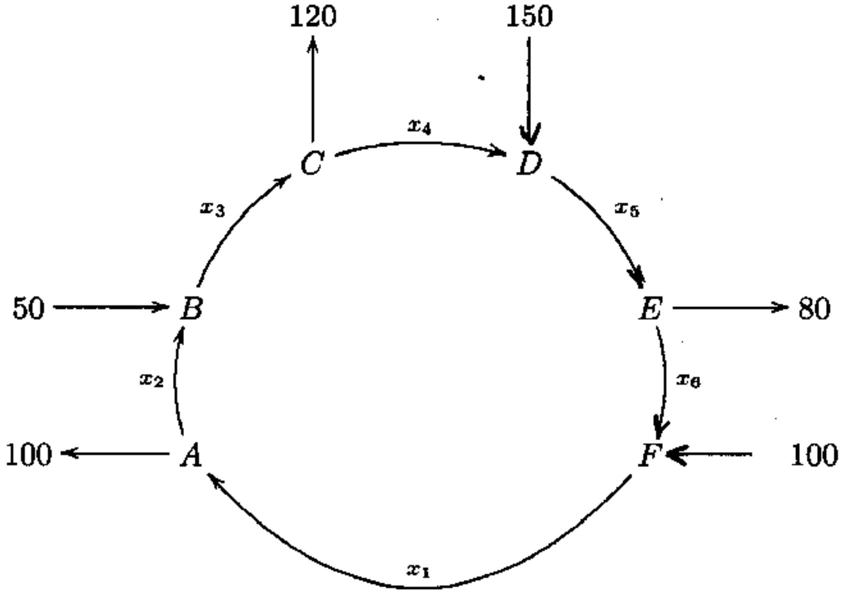
$$\text{rank}[A|b] = \begin{cases} 3 & \text{if } c \neq 0 \\ 3 & \text{if } c = 0 \;\&\; a \neq 0 \\ 2 & \text{if } c = 0 \;\&\; a = 0 \end{cases}$$

(i) The system will have a unique soln $\Leftrightarrow c \neq 0$ (rank A = rank(A|b) = 3)

(ii) The system will have ∞ly many solns when $c = 0$ & $a = 0$
(rank A = rank (A|b) = 2)

(iii) The system will be inconsistent if $c = 0$ and $a \neq 0$
(rank A = 2 < rank [A|b]

11. (b) Consider the network of streets and intersections below, known as a 'roundabout'. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave the intersections during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



Write down a system of equations, **together with all constraints**, that will determine all possible traffic flows. **DO NOT SOLVE THIS SYSTEM.**

Constraints: $x_i \geq 0$ $i=1,..,6$

& $x_i \in \mathbb{Z}$

System:

FLOW IN = FLOW OUT

A $x_1 = 100 + x_2$

B $x_2 + 50 = x_3$

C $x_3 = 120 + x_4$

D $x_4 + 150 = x_5$

E $x_5 = x_6 + 80$

F $x_6 + 100 = x_1$

12. Let $U = \{(x, y, z) \in \mathbf{R}^3 \mid x + 2y + z = 0\}$.

a) Find a basis of U and give the dimension of U.

b) Find an orthogonal basis of U.

c) Find the best approximation to $(1, 1, 0)$ by vectors in U.

a) $\begin{bmatrix} 1 & \overset{s}{2} & \overset{t}{1} & | & 0 \end{bmatrix}$ $\quad \begin{aligned} x &= -2s - t \\ y &= s \\ z &= t \end{aligned}$; $s, t \in \mathbb{R}$ $\therefore$ $\{\overset{v_1}{(-2,1,0)}, \overset{v_2}{(-1,0,1)}\}$ is a basis for U. $\therefore$ $\dim U = 2$

b) Let $u_1 = v_1$

$$u_2 = v_2 - \operatorname{proj}_{u_1} v_2 = v_2 - \frac{v_1 \cdot v_2}{\|v_1\|^2} v_1$$

$$= (-1,0,1) - \frac{2}{5}(-2,1,0)$$

$$= \left(-\frac{1}{5}, \frac{-2}{5}, 1\right) = \frac{1}{5}(-1,-2,5)$$

(check: ① $u_2 \cdot u_1 = \frac{1}{5}(-1,-2,5) \cdot (-2,1,0) = 0$ ✓; ② $-\frac{1}{5} + 2\left(\frac{-2}{5}\right) + 1 = 0$ $\therefore u_2 \in U$ ✓

$\therefore \{u_1, u_2\}$ is an orthogonal basis of U.

Note: $\|u_2\|^2 = \frac{1}{25}(1+4+25) = \frac{30}{25} = \frac{6}{5}$

c) The best approximation will be

$$\operatorname{proj}_U(1,1,0) = \frac{u_1 \cdot (1,1,0)}{\|u_1\|^2} u_1 + \frac{u_2 \cdot (1,1,0)}{\|u_2\|^2} u_2$$

$$= \frac{-1}{5}(-2,1,0) + \frac{1}{5} \frac{(-3)}{26/\cancel{5}} \cdot \cancel{\frac{1}{5}}(-1,-2,5)$$

$$= \left(\frac{2}{5}, \frac{-1}{5}, 0\right) - \frac{1}{10}(-1,-2,5)$$

$$= \left(\frac{5}{10}, 0, \frac{-5}{10}\right) = \left(\frac{1}{2}, 0, -\frac{1}{2}\right).$$

[Check: ① $\frac{1}{2} + 2(0) + \left(-\frac{1}{2}\right) = 0$
$\therefore \operatorname{proj}_U(1,1,0) \in U$

② $(1,1,0) - \left(\frac{1}{2}, 0, -\frac{1}{2}\right)$
$= \left(\frac{1}{2}, 1, \frac{1}{2}\right)$ is parallel to the normal of U and so is perpendicular to every vector in U]

13. Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

a) Find the characteristic polynomial $\det(A - xI)$ of A, and deduce that the eigenvalues of A are 1 and 3.

b) Find a basis of $E_1 = \{x \in \mathbf{R}^3 \mid Ax = x\}$.

b) Find a basis of $E_3 = \{x \in \mathbf{R}^3 \mid Ax = 3x\}$.

d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Explain why your choice of P is invertible.

a) $|A - xI| = \begin{vmatrix} 2-x & -1 & 0 \\ -1 & 2-x & 0 \\ 0 & 0 & 3-x \end{vmatrix} \overset{\text{col } 3}{=} (3-x) \begin{vmatrix} 2-x & -1 \\ -1 & 2-x \end{vmatrix} = (3-x)(x^2-4x+4-1)$

$= (3-x)(x^2-4x+3)$

$= (3-x)(x-3)(x-1)$

∴ $\lambda = 3, 1$ are evals.

b) $E_1 = \ker(A - I) = \ker \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & \overset{s}{-1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ∴ $\begin{matrix} x = s \\ y = s \\ z = 0 \end{matrix}$; $s \in \mathbb{R}$

∴ $\{(1,1,0)\}$ is a basis for E_1

c) $E_3 = \ker(A - 3I) = \ker \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & \overset{s}{1} & \overset{t}{0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x = -s \\ y = s \\ z = t \end{matrix}$; $s, t \in \mathbb{R}$

∴ $\{(-1,1,0), (0,0,1)\}$ is a basis for E_3

d) Let $P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $\det P = 1.(2) = 2 \neq 0$ so P is inv.

Moreover $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$.

14. Let $u = (1, 0, 1)$ and define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(v) = u \times v, \qquad v \in \mathbf{R}^3,$$

where "×" denotes the cross product.

a) Show that $S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -y \\ x - z \\ y \end{bmatrix}$

b) Find the standard matrix of S.

c) Find a basis for im S and describe it geometrically.

d) Find $\dim \ker S$.

a) $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \vec{k} \\ 1 & 0 & 1 \\ x & y & z \end{vmatrix} = (-y, -(z-x), y) \quad \therefore S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -y \\ x-z \\ y \end{bmatrix}$, as req'd.

b) $S(v) = Av$ where $A = [S(e_1) \;\; S(e_2) \;\; S(e_3)]$, $S(e_i)$ as columns,

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

c) $\operatorname{im} S = \operatorname{col} A$; $A \sim \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \{(0,1,0), (-1,0,1)\}$ is

a basis for im S. Thus, Im S is the plane through O with normal vector $(0,1,0) \times (-1,0,1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1, -0, 1) = (1,0,1)$ $(= u)$

d) $\dim \ker S + \dim \operatorname{im} S = \dim \mathbb{R}^3 = 3 \quad \therefore \dim \ker S = 3 - \dim \operatorname{im} S$
$= 3 - 2$
$= 1.$

15. State whether the following are true or false. If true, explain why, if false, give an explicit counter-example to illustrate.

a) A basis of a finite-dimensional vector space is a spanning set which has the largest number of elements possible.

This is false. e.g. let $V = \mathbb{R}^2$. Then $\{(1,0),(0,1),(1,1)\}$ spans $\mathbb{R}^2$ but is not a basis. Since $\dim \mathbb{R}^2 = 2$, any set with more than 2 vectors in it will be l.d., and hence not a basis. There are spanning sets with arbitrarily large numbers of vectors: eg $\{(1,0),(0,1),(0,2),(0,3), \dots ,(0,n)\}$.

• A spanning set with the least number of elements possible will be a basis.

b) Every diagonalizable $n \times n$ matrix is invertible.

This is false : e.g. $A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonal (and hence diagonalizable) but is not invertible, as $\operatorname{rank} A = 1 < 2$.

15c) If $v \in \mathbf{R}^n$ satisfies $v \cdot w$ for all $w \in \mathbf{R}^n$, then $v = 0$.

This is True : v must satisfy $v \cdot v = 0$

ie. $\|v\|^2 = 0$

But this is equivalent to $v = 0$.

15d) Let $T : \mathbf{R}^4 \to \mathbf{R}^6$ be a linear transformation such that $\ker T = \{0\}$. If $\{v_1, v_2, v_3\}$ is linearly independent in $\mathbf{R}^4$, then $\{T(v_1), T(v_2), T(v_3)\}$ is linearly independent in $\mathbf{R}^6$.

This is True. Suppose $a\,T(v_1) + b\,T(v_2) + c\,T(v_3) = 0$

Then (since T is linear) $T(av_1 + bv_2 + cv_3) = 0$.

That is, $av_1 + bv_2 + cv_3 \in \ker T = \{0\}$, So

$$a\,v_1 + bv_2 + cv_3 = 0$$

But $\{v_1, v_2, v_3\}$ is l.i., so $a = b = c = 0$.

Hence $\{Tv_1, Tv_2, Tv_3\}$ is also l.i.

15e) **(Bonus: 1.5 marks)** If 0 is the only eigenvalue of a 2×2 matrix A, then $A = 0$.

This is false : e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = \lambda^2 = 0 \iff \lambda = 0$.

So $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has zero as its only eval, but is not the zero matrix.