Mat1341A.F04 final Solutions   12 mins MC

1. A homogeneous linear system of 2004 equations in 244 unknowns...
always consistent

A. can be inconsistent. ✗

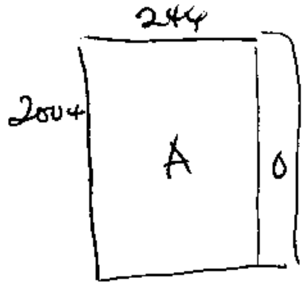
B. is never inconsistent and will never have a unique solution.

C. is never inconsistent and always has a unique solution.

D. is never inconsistent and may have a unique solution.

E. is never inconsistent and has 1760 parameters in the solution.

F. has 1760 solutions or less. (could have only many)



rank $A \le 244$;
rank A could be 244
$\therefore$ could have unique soln

\#parameters $= 244 - \operatorname{rank} A \le 244$.

2. Compute the determinant $\begin{vmatrix} 0 & 0 & 0 & 7 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix} \overset{col4}{=} -7 \cdot \begin{vmatrix} 3 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 \end{vmatrix}$

A. -135

B. -105

C. 165

D. -205

E. -175

F. 225

$\overset{col4}{=} -7 \cdot (-1) \begin{vmatrix} 3 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 5 & 0 \end{vmatrix}$

$\overset{row3}{=} 7 \cdot (-5) \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$

$= -7 \cdot 25$

$= -175$

3. The inverse of the matrix $E = \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

$\left[\begin{array}{ccc|ccc} 1 & 0 & -x & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

A. $\begin{bmatrix} 1 & 0 & \frac{1}{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ if $x \neq 0$

B. $\begin{bmatrix} 1 & 0 & -\frac{1}{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ if $x \neq 0$

C. $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} x & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -x \end{bmatrix}$

F. $\begin{bmatrix} x & 0 & -1 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$

4. The dimension of $S = \{A \in M_{22} \mid A = A^t\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in M_{22} \right\}$

$= \text{span}\left\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{M_3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

& $aM_1 + bM_2 + dM_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leftrightarrow a = b = d = 0$

$\therefore \dim S = 3$

5. Which of the following are bases for $\mathbf{R}^4$?

(1) $\{(1,0,0,0),(4,\ 2,\ 0,0),\ (1,\ 4,\ 0,0),\ (5,\ 6,\ 1,0)\}$

(2) $\{(0,-1,\ 2,\ 3),\ (0,3,\ 3,\ 2),(4,\ 2,\ 0,0)\}$

(3) $\{(-1,\ 3,\ -5,1),\ (1,\ -2,\ 4,2),\ (2,\ 0,\ 4,3),\ (5,\ 1,\ 9,4),(4,\ 2,\ 0,0)\}$

A. All three
B. (1) only
C. (2) only
D. (1) and (2)
E. (2) and (3)
F. None of them is a basis of $\mathbf{R}^4$.

Since $\dim \mathbb{R}^4 = 4$, the only possibility is (1):

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 5 & 6 & 1 & 0 \end{vmatrix} = 0 \text{ since last column is zero}$$

$\therefore$ the vectors (rows of this matrix) are l.d., so (1) is not a basis

6. If A is an $n \times 2$ matrix and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then the second column of the matrix AB is

A. not defined unless $n = 2$.
B. the same as the second column of A.
C. the same as the second column of B.
D. the same as the first column of A.
E. the same as the first column of B.
F. the sum of the first and the second column of A.

$A = [C_1\ C_2]$ $\quad C_i = i^{\text{th}}$ col of A

$$AB = [C_1\ C_2]\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= [C_1 + C_2 \quad C_1]$$

4 | 5

7. Let A be a 4×5 matrix such that rank $A = 4$. Then,

A. $\ker A = \{0\}$, row $A = \mathbf{R}^5$
B. $\ker A = \{0\}$, col $A = \mathbf{R}^4$
C. $\dim \ker A = 4$, $\dim \text{col } A = 1$
D. $\dim \ker A = 1$, $\dim \text{col } A = 4$
E. $\dim \ker A = 1$, $\dim \text{row } A = 1$
F. $\dim \ker A = 4$, $\dim \text{row } A = 4$

dim ker A + rank A = # cols = 5

$\therefore$ dim ker A = 5 − 4 = 1

and 4 = rank A = dim col A = dim row A

8. Which of the following are subspaces of $\mathbf{R}^3$?

(1) $\{(x,\ y,\ z) \mid x - y + z = 0\}$ is a plane through origin $\therefore$ is a s.s. of $\mathbb{R}^3$
(2) $\{(x,\ y,\ z) \mid xyz = 0\}$
(3) $\{(x,\ y,\ z) \mid 2x = 5z = 6y\}$ is a line through origin $\therefore$ is a s.s. of $\mathbb{R}^3$
(4) $\{(x,\ y,\ z) \mid x = y - 1 = z\}$ is a line not passing through the origin $\therefore$ isn't a s.s.

A. (1) and (2)
B. (1) and (3) (since (1) & (2) & (3) isn't a possibility)
C. (2) and (4)
D. (2) and (3)
E. (1), (3) and (4)
F. (3) and (4)

note: (2) is not a s.s. since

$u = (1,0,0) \in (2)$

$v = (0,1,1) \in (2)$

but $u + v = (1,1,1) \notin (2)$.

9. A matrix which **does not have** the same eigenvalues as $G \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ is

A. $\begin{bmatrix} 3 & 2 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 2 \end{bmatrix}$ B. $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ 9 & 2 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 8 & 2 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ F. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

The eigenvalues of G are 1, 2, 3 (as G is upper $\triangle$ular, $\det(G - xI) = (1-x)(2-x)(3-x)$). Each of other matrices is upper or lower $\triangle$ular, so evals are diagonal entries; only F has different eigenvalues (1, & 3)

10. Which of the following are linearly independent in $\mathbf{F}[0, 2\pi] = \{f \mid f : [0, 2\pi] \to \mathbf{R}\}$?

$S = \{\cos^2 x, \sin^2 x\}$

$T = \{1, \sin^2 x\}$

$U = \{1, \cos^2 x, 3\sin^2 x\}$ $1 = \cos^2 x + \frac{1}{3} \cdot 3\sin^2 x$ $\therefore$ l.d

$V = \{\sin 2x, 2\sin x \cos x\}$ $\sin 2x = 2\sin x \cos x$ $\therefore$ l.d.

A. T and V.
B. T and U.
C. S and T. since all others include either U or V.
D. S and V.
E. S, U and V.
F. S, U and T.

11. (a) Consider the linear system

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0 \\ 3x & + & 2y & + & kz & = & 0 \\ 4x & + & 9y & + & 3z & = & 0 \end{array}$$

Find all k so that this system has

(i) a unique solution,
(ii) infinitely many solutions, and
(iii) no solutions.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & 2 & k & 0 \\ 4 & 9 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -4 & k-3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k-7 & 0 \end{array}\right] \ast$$

$\frac{1}{2}$

so $\quad \text{rank}A = \text{rank}[A|b] = 3 \iff k \neq 7$ $\qquad 1 = \frac{1}{2} + \frac{1}{2}$

i) $\quad \text{rank}A = \text{rank}[A|b] < 3 \iff k = 7$ $\qquad 1 = \frac{1}{2} + \frac{1}{2}$

iii) This is a homogeneous system and so is always consistent.
(ie there are no values of k for which it is inconsistent.) $\frac{1}{2}$

(consistent with $\ast$)

11. (b) A Norweigan Blue parrot is advised by a nutritionist to take 14 units of vitamin A, 15 units of vitamin D and 29 units of vitamin E each day. The parrot can choose from three brands of pill (say) I, II and III, and the amount of each vitamin in every pill of the various brands is given below:

	I	II	III
vitamin A	2	1	1
vitamin D	3	3	0
vitamin E	5	4	1

Write down a system of equations, together with all constraints, that will determine all possible combinations of numbers of pills of each brand that will provide exactly the required daily amounts of vitamins for the parrot.

Don't forget to define your variables but DO NOT SOLVE THIS SYSTEM.

Let x = # pills of type I taken per day

$\frac{1}{2}$ y = " II " ; then $x, y, z \geq 0$ $\frac{1}{2}$

z = " III " $\to x, y, z \in \mathbb{Z}$ $\frac{1}{2}$

To exactly fulfill the requirements,

vitamin A: $14 = 2x + y + z$ $\frac{1}{2}$

D $15 = 3x + 3y + 0z$ $\frac{1}{2}$

E $29 = 5x + 4y + z$ $\frac{1}{2}$

12. Let $U = \{(x,y,z) \in \mathbf{R}^3 \mid x+y+z=0\}$.

$2\frac{1}{2}$ a) Find a basis of U and give the dimension of U.

$1\frac{1}{2}$ b) Find an orthogonal basis of U.

2 c) Find the best approximation to $(1,0,1)$ by vectors in U.

a) $\begin{array}{cccc} & s & t & \\ \end{array}$ $[1 \quad 1 \quad 1 \mid 0]$ $\quad x = -s-t, \quad y = s, \quad z = t$ $\quad (1) \quad ; s,t \in \mathbb{R}$

$2\frac{1}{2}$

(1) $\therefore \{\underset{v_1}{(-1,1,0)}, \underset{v_2}{(-1,0,1)}\}$ is a basis of U.

$\left(\frac{1}{2}\right)$ $\therefore \dim U = 2$

b) Apply G-S: $\quad u_1 = (-1,1,0) = v_1$

$\left(\frac{1}{2}\right)$ $\quad u_2 = (-1,0,1) - \dfrac{u_1 \cdot v_2}{\|u_1\|^2} u_1$

(any) method

$$= (-1,0,1) - \frac{1}{2}(-1,1,0) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1\right)$$

$\therefore \{(-1,1,0), (-\tfrac{1}{2}, -\tfrac{1}{2}, 1)\}$ is an orthogonal basis of U

(or $\{\underset{w_1}{(-1,1,0)}, \underset{w_2}{(1,1,-2)}\}$, e.g.) $\quad \left(\frac{1}{2}\right)$ – orthogonal $\quad \left(\frac{1}{2}\right)$ – both belong to U.

c) The best approx is $\mathrm{proj}_U (1,0,1) = \left(\dfrac{(1,0,1)\cdot w_1}{\|w_1\|^2}\right) w_1 + \dfrac{(1,0,1)\cdot w_2}{\|w_2\|^2} w_2$ $\quad (1)$

$$= \frac{-1}{2} \cdot (-1,1,0) + \frac{(-1)}{6} \cdot (1,1,-2) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0\right) + \left(-\tfrac{1}{6}, -\tfrac{1}{6}, \tfrac{1}{3}\right)$$

$$= \left(\tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right)$$

$\left(\frac{1}{2}\right) \in U$

$+$

$\left(\frac{1}{2}\right)$ if also correct

13. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

$1\frac{1}{2}$ a) Find the characteristic polynomial $\det(A - xI)$ of A, and deduce that the eigenvalues of A are 1 and 4.

$1\frac{1}{2}$ b) Find a basis of $E_1 = \{x \in \mathbf{R}^3 \mid Ax = x\}$.

1 c) Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 4.

2 d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Explain why your choice of P is invertible.

a) $|A - xI| = \begin{vmatrix} 2-x & 1 & 1 \\ 1 & 2-x & 1 \\ 1 & 1 & 2-x \end{vmatrix} = \begin{vmatrix} 2-x & 1 & 1 \\ 1 & 2-x & 1 \\ -1+x & 0 & 1-x \end{vmatrix} = \begin{vmatrix} 2-x & 1 & 3-x \\ 1 & 2-x & 2 \\ -1+x & 0 & 0 \end{vmatrix}$

$\overset{\text{row } 3}{=} (x-1) \begin{vmatrix} 1 & 3-x \\ 2-x & 2 \end{vmatrix} = (x-1)\left(2 - (x-3)(x-2)\right) = (x-1)\{2 - x^2 + 5x - 6\}$

$= (x-1)\{-x^2 + 5x - 4\}$

$= (x-1)(-x+1)(x-4)$

$= (x-1)^2(4-x)$ (1)

$\therefore |A - xI| = 0 \Leftrightarrow x = 1, 4.$ ($\frac{1}{2}$) ← "~~requires~~ for the deduction"

So eval are 1, 4

b) $E_1 = \ker(A - I)$: $[A - I \mid 0] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \begin{array}{cccc} & s & t & \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \left|\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] \end{array}$ $\begin{array}{l} x = -s - t \\ y = s \\ z = t \end{array}$ ($\frac{1}{2}$)

$\therefore \{(-1, 1, 0), (-1, 0, 1)\}$ is a basis for E_1 (1)

c) $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\therefore \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an evec of A with eval 4

d) Set $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (1) Then $\det P = \begin{vmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$ $\therefore P$ is inv ($\frac{1}{2}$)

Since $AP = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, which is diagonal

14. Define a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 0 \\ z - x \end{bmatrix}$$

1 a) Find the standard matrix of T.

2 b) Find a basis of $\ker T$.

2½ c) Find the dimension of $\operatorname{im} T$, 1 and give a complete geometric description of $\operatorname{im} T$. $1\frac{1}{2}$

1 d) Show that if $u \in \operatorname{im} T$, then $T(u) = 2u$.

a) $A = [Te_1 \quad Te_2 \quad Te_3] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

b) $\ker T = \ker A$: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim \begin{array}{c} \quad\; s \quad t \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ $\begin{array}{l} x = t \\ y = s \\ z = t \end{array}$ 1 $\therefore \{(1,0,1), (0,1,0$ 1

is a basis of $\ker T$

c) $\dim \operatorname{im} T = \dim \operatorname{col} A = \operatorname{rank} A = 1.$ 1 Now, $\operatorname{im} T = \operatorname{col} A = \operatorname{span}\{(1,0,-1)$ from calculations in (b). Hence $\operatorname{im} T$ is the line through 0 in $\mathbb{R}^3$ with direction $(1,0,-1)$.

d) If $u \in \operatorname{im} T$, then $u = s(1,0,-1)$, $s \in \mathbb{R}$, so

$$Tu = T\begin{bmatrix} s \\ 0 \\ -s \end{bmatrix} = \begin{bmatrix} 2s \\ 0 \\ -2s \end{bmatrix} = 2\begin{bmatrix} s \\ 0 \\ -s \end{bmatrix} = 2u.$$

15. a) State whether the following are true or false. If true, explain why, if false, give a numerical example to illustrate.

i) If W is a subspace of $\mathbf{R}^3$ and u and v are vectors such that $u+v \in W$, then u and v are also in W.

This is false e.g. Let $W = \{(0,0,0)\}$, $u=(1,0,0)$ & $v=(-1,0,0)$.
Then W is a subspace of $\mathbb{R}^3$, $u \notin W$, $v \notin W$ but $u+v=(0,0,0) \in W$.

ii) if A is a 13 by 13 matrix such that $A^7 = 0$, then A is not invertible. TRUE

If $A^7=0$, then $0=\det(A^7) = (\det A)^7$ $\therefore$ $\det A=0$
Hence, A is not invertible.

V \\

iii) $\{A \in \mathbf{M}_{22} \mid \det A = 0\}$ is a subspace of $\mathbf{M}_{22}$. This is FALSE.

eg: $M_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in V$ and $M_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in V$ but $M_1+M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin V$,
($\det M_1=0$) ($\det M_2=0$)
since $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$

(Please note that 15(b) is on the next page)

15b) Let A be a real $n \times n$ matrix. Give another 3 different statements equivalent to

"$Ax = 0$ has a unique solution",

(I) A is invertible

(II) $\det A \neq 0$

(III) $\operatorname{rank} A = n$

plus many others: see your notes.