1341A F12 Ass. 3 Solns

1. Suppose $v_1 = (1,2,2,4), v_2 = (1,2,4,4), v_3 = (2,4,6,8), v_4 = (2,4,5,8)$ and let

$$V = \text{span}\{v_1, v_2, v_3, v_4\}.$$

Find a basis for V which is a <u>**subset**</u> of the given spanning set $\{v_1, v_2, v_3, v_4\}$

We use the column space algorithm. set $A = [v_1\ v_2\ v_3\ v_4]$ then col $A = V$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 6 & 5 \\ 4 & 4 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence $\{v_1, v_2\}$ is a basis for col$A = V$

(one more step and we'd be in RE form)

ANSWER $\boxed{\{v_1, v_2\}}$

3. If $\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = 3$, find $\begin{vmatrix} b & f-2n & n & 5j-b \\ a & e-2m & m & 5i-a \\ c & g-2o & o & 5k-c \\ d & h-2p & p & 5l-d \end{vmatrix}$.

$$\underset{\substack{2C_3+C_2 \to C_2 \\ C_1+C_4 \to C_4}}{=} \begin{vmatrix} b & f & n & 5j \\ a & e & m & 5i \\ c & g & o & 5k \\ d & h & p & 5l \end{vmatrix}$$

$$\underset{(B \to B^t)}{=} \begin{vmatrix} b & a & c & d \\ f & e & g & h \\ n & m & o & p \\ 5j & 5i & 5k & 5l \end{vmatrix} = 5 \begin{vmatrix} b & a & c & d \\ f & e & g & h \\ n & m & o & p \\ j & i & k & l \end{vmatrix} \underset{C_1 \leftrightarrow C_2}{=} -5 \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ m & n & o & p \\ i & j & k & l \end{vmatrix}$$

$$\underset{R_3 \leftrightarrow R_4}{=} -(-5) \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = 5 \cdot 3 = 15$$

ANSWER $\boxed{15}$

2. One of the eigenvalues of the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 0 & -2 \\ -6 & 6 & -6 \end{bmatrix}$ is -1. The other two eigenvalues of A are:

A. 2 and -1
B. 1 and 3
C. 0 and 1
D. 2 and 3
E. -3 and 0
F. -1 and 3

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ -1 & -\lambda & -2 \\ -6 & 6 & -6-\lambda \end{vmatrix} \overset{C_2+C_1 \to C_1}{=} \begin{vmatrix} -\lambda-1 & -3 & 1 \\ -\lambda-1 & -\lambda & -2 \\ 0 & 6 & -6-\lambda \end{vmatrix}$$

$$= -(1+\lambda) \begin{vmatrix} 1 & -3 & 1 \\ 1 & -\lambda & -2 \\ 0 & 6 & -6-\lambda \end{vmatrix} \overset{-R_1+R_2 \to R_2}{=} -(1+\lambda) \begin{vmatrix} 1 & -3 & 1 \\ 0 & 3-\lambda & -3 \\ 0 & 6 & -6-\lambda \end{vmatrix} \overset{\text{col } 1}{=} -(1+\lambda) \begin{vmatrix} 3-\lambda & -3 \\ 6 & -6-\lambda \end{vmatrix}$$

$$= -(1+\lambda)\{(\lambda-3)(\lambda+6)+18\} = -(1+\lambda)\{\lambda^2+3\lambda\} = -(1+\lambda)\lambda(\lambda+3)$$

$\therefore$ evals are $\lambda = -1$, $\lambda = 0$, $\lambda = -3$

ANSWER $\boxed{0 \text{ and } -3}$

4. Suppose $V = \text{span}\{\overset{v_1}{(1,1,1,1)}, \overset{v_2}{(1,-1,-1,1)}, \overset{v_3}{(1,1,1,2)},\}$. ($\ast$ Remark: $V = \{(x,y,z,t) \mid y-z=0\}$)

(How did I obtain this?)

a) Find an orthogonal basis for V.

b) Find a formula for the orthogonal projection $\text{proj}_V(x,y,z,t)$ of any vector $(x,y,z,t) \in \mathbf{R}^4$ onto V.

c) Find the best approximation (call it v) in V to the vector $w = (0,1,0,1)$, and compute the approximation error $\|w-v\|$.

a) We apply Gram-Schmidt: First note that $v_1 \cdot v_2 = 0$, so $w_1 = v_1$, $w_2 = v_2$. Then

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2}{\|w_2\|} w_2$$

$$= (1,1,1,2) - \frac{5}{4}(1,1,1,1) - \frac{1}{4}(1,-1,-1,1)$$

$$= \left(-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right) \qquad (\underline{\text{Check}}: w_3 \cdot v_1 = w_3 \cdot v_2 = 0 \checkmark)$$

Hence $\{\overset{w_1}{(1,1,1,1)}, \overset{w_2}{(1,-1,-1,1)}, \overset{\tilde{w}_3}{(-1,0,0,1)}\}$ is an orthogonal basis for V. (We may re-scale w_3 to simplify later calculations)

(Note: If we first simplified our spanning set by using row-reduction i.e. $V = \text{row}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \text{row}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{row}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then apply G-S to the non-zero rows of the last matrix further simplifies the computation. This is <u>not</u> necessary, but can be helpful.)

b) For $v = (x,y,z,t)$, $\text{proj}_V v = \dfrac{v \cdot w_1}{\|w_1\|^2} w_1 + \dfrac{v \cdot w_2}{\|w_2\|^2} w_2 + \dfrac{v \cdot \tilde{w}_3}{\|\tilde{w}_3\|^2} \tilde{w}_3$

$$= \frac{(x+y+z+t)}{4}(1,1,1,1) + \frac{(x-y-z+t)}{4}(1,-1,-1,1) + \frac{(-x+t)}{2}(-1,0,0,1)$$

$= \left(x, \frac{y+z}{2}, \frac{y+z}{2}, t\right)$. (Check: Using $\ast$ above, this vector <u>is</u> (at least) in W).

c) The best approximation $v = \text{proj}_V(0,1,0,1) = (0, \tfrac{1}{2}, \tfrac{1}{2}, 1)$, by (c)

The error in approximation is therefore $\|v - (0, \tfrac{1}{2}, \tfrac{1}{2}, 1)\|$

$$= \|(0, \tfrac{1}{2}, -\tfrac{1}{2}, 0)\| = \sqrt{\tfrac{1}{2}} = \frac{\sqrt{2}}{2}.$$

5. Suppose

$$A = \begin{bmatrix} 3 & 0 & 4 \\ 0 & -1 & 0 \\ 4 & 0 & 3 \end{bmatrix}.$$

a) Compute the cubic polynomial $\det(A - \lambda I_3)$, then factorize it, and hence find the eigenvalues of A. Note: $|A - \lambda I_3| = \begin{vmatrix} 3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ 4 & 0 & 3-\lambda \end{vmatrix} \overset{col 2}{=} -(1+\lambda) \begin{vmatrix} 3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix}$

$= -(1+\lambda)\{ \underbrace{\lambda^2 - 6\lambda + 9 - 16}_{(\lambda^2-6\lambda-7)} \} = -(1+\lambda)(\lambda+1)(\lambda-7) = (1+\lambda)^2(7-\lambda).$

Hence $|A - \lambda I_3| = 0 \Leftrightarrow \lambda = -1$ or $\lambda = 7$. Thus the eigenvalues of A are -1 and 7.

b) Find a basis for $\ker(A + I_3) = \ker \begin{bmatrix} 4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix} = \ker \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{pmatrix} x = -t \\ y = s \\ z = t \end{pmatrix}$

$= \{(-t, s, t) \mid s, t \in \mathbb{R}\}$

$= \text{span}\{(0,1,0), (-1,0,1)\}$. By a theorem in class, this algorithm yields the basis $\{(0,1,0), (-1,0,1)\}$ for $\ker(A+I)$.

(Or: you can see these span, and they are orthogonal and hence l.i.)

5 c). Find a basis for $\ker(A - 7I_3)$ $= \ker \begin{bmatrix} -4 & 0 & 4 \\ 0 & -8 & 0 \\ 4 & 0 & -4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x = s \\ y = 0 \\ z = s \end{matrix}$

$= \{(s,0,s) \mid s \in \mathbb{R}\}$
$= \mathrm{Span}\{(1,0,1)\}$. Hence $\{(1,0,1)\}$ is a basis for $\ker(A-7I_3)$.

d) Collect all the basis vectors you found in (b) and (c), and construct a matrix P with these vectors as columns. Is your matrix P invertible?

Hence $P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, which shows rank $P = 3$. Hence P is invertible.

OR $\det P = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} \overset{\text{row 2}}{=} - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.$

OR (We learned this Thursday 29-Nov)

$\dim E_{-1} + \dim E_7 = 2 + 1 = 3 = \dim \mathbb{R}^3$.

Hence, the union of the bases in (b) & (c) is a basis of $\mathbb{R}^3$, and as these vectors are the cols. of P, P is invertible.