MATH1005E — Test 1 —- 11:35–12:25, Jan. 30 2013

Total: 20 marks

Question 1. Consider the equation $xy + y^2 - x^2y' = 0$.

(a) Solve it as homogeneous equation.[4]

(b) Solve it as a Bernoulli equation[4]

Solution:
(a) Divide by x^2, so $\frac{y}{x} + (\frac{y}{x})^2 - y' = 0 \to y' = \frac{y}{x} + (\frac{y}{x})^2$ and $u = \frac{y}{x} \to y = ux \to y' = u + u'x = u + u^2 \to u'x = u^2$
$\frac{u'}{u^2} = \frac{1}{x} \to \int \frac{1}{u^2}du = \int \frac{1}{x}dx \to -u^{-1} = \ln|x| + c \to u = \frac{-1}{\ln|x|+c} \to y = \frac{-x}{\ln|x|+c}$

(b) Divide by $-x^2$, then $y' - \frac{1}{x}y = \frac{y^2}{x^2}$, so $\alpha = 2$ and $u = y^{-1} \to y = u^{-1} \to y' = -u^{-2}u'$
$-u^{-2}u' - \frac{1}{x}u^{-1} = \frac{u^{-2}}{x^2}$ multiply by $u^2 \to u' + \frac{1}{x}u = -\frac{1}{x^2}$ is linear, where
$I(x) = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$ then
$u = \frac{1}{x}\int(-\frac{1}{x^2})(x)dx = \frac{1}{x}\int -\frac{1}{x}dx = \frac{1}{x}(-\ln|x| + c) \to y = u^{-1} \to y = \frac{-x}{\ln|x|-c}$

Question 2. Solve the initial-value problem [4]

$$2x^2y' - 2xy = -y^3\sin(x), \qquad y(\frac{\pi}{2}) = \pi$$

Solution:
It is Bernoulli equation that $\alpha = 3$ so $u = y^{-2} \to y = u^{-1/2} \to y' = -\frac{1}{2}u^{-\frac{3}{2}}u',\ y^3 = u^{-3/2}$
$-x^2u^{-3/2}u' - 2xu^{-1/2} = -u^{-3/2}\sin(x)$, multiply by $-u^{3/2}$, then $x^2u' + 2xu = \sin(x)$ is linear and
$u' + \frac{2}{x}u = \frac{\sin(x)}{x^2}$, and $I(x) = e^{\int p(x)dx} = e^{\int \frac{2}{x}} = x^2$
$u = \frac{1}{x^2}\int \frac{\sin(x)}{x^2}(x^2)dx = \frac{1}{x^2}\int \sin(x) = \frac{1}{x^2}(-\cos(x) + c) \to u = \frac{c-\cos(x)}{x^2}$
$y = u^{-1/2} = \pm\frac{x}{\sqrt{c-\cos(x)}}$ initial-condition is $y(\frac{\pi}{2}) = \pi$
$\pi = \frac{\pi/2}{\sqrt{c-0}} \to c = \frac{1}{4}$

Question 3. Find the general solution of the equation $y' = 2x(y^2 + 1)$ and its orthogonal trajectories.[4]

Solution:
$\frac{y'}{y^2+1} = 2x \to \int \frac{1}{y^2+1}dy = \int 2xdx \to \tan^{-1}(y) = x^2 + c \to y = \tan(x^2 + c)$ is the general solution and its orthogonal trajectories has $y' = \frac{-1}{2x(y^2+1)}$, then
$(y^2 + 1)y' = \frac{-1}{2x} \to \int(y^2 + 1)dy = \int \frac{-1}{2x}dx \to \frac{y^3}{3} + y = \frac{-1}{2}\ln|x| + c$

Question 4. Let $f(x,y) = x^4+3x^2y^2+y^4$ and $y(x) = x^3+6x^2-x$. Determine $\frac{d}{dx}f(x,y(x))$.[4]

Solution:

$$\frac{d}{dx}f(x,y(x)) = f_x + f_y\frac{dy}{dx} = 4x^3 + 6xy^2 + (6x^2y + 4y^3)(3x^2 + 12x - 1)$$