Key for: 1104-B, E

Carleton University, School of Math & Stats
Math 1104-E [F13] — TEST 1 [40 points]
ALERT: What is your TUTORIAL SECTION ? → [ ]

Dr. ℜJ Cova

§1. Solve the linear system $2x - 3y + z - 1 = 1$, $2z - x = 0$, $3x - 3y = z + 1$ using (a)Gauss-Jordan elimination (RREF) and also, if possible, (b) by the method of inverses. (c)Provide a geometrical interpretation of your result. Please show your work and/or explanations. (15 marks: 10, 3, 2)

(a) $\left(\begin{array}{ccc|c} 2 & -3 & 1 & 2 \\ -1 & 0 & 2 & 0 \\ 3 & -3 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -3 & 5 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right) \to$ INCONSISTENT, NO SOLUTION!

5 marks

5 marks

(b) Not possible 'cos $A^{-1} \nexists$, ie, $|A| = 0$. 3 marks

(c) The three ~~planes~~ do not intersect at all!

$\Pi_j \cancel{\cap}$

2 marks

§2. (6 marks: 2 each) A certain linear system has the augmented matrix $\left(\begin{array}{cc|c} 1 & h & 2 \\ 0 & 2-h & 1 \end{array}\right)$. Determine the values of h for which the system is (a)consistent or (b)inconsistent. (c)In the first case does the system possess only one solution or infinitely many solutions? Explain in detail.

(a) Consistent for $h \neq 2$ since we get $\left(\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & \beta \end{array}\right)$

$\to \begin{cases} x = \alpha \equiv (3-2h)/(2-h) \\ y = \beta \equiv 1/(2-h) \end{cases}$, clearly a UNIQUE solution. (c)

(b) Inconsistent for $h = 2$ because $\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 1 \end{array}\right) \to$ $0 = 1$, which is impossible.

§3. Consider the linear transformation $T < x, y, z > = < -2x+3y-z, x-2z, -3x+3y+z >$. (*a*) Obtain A, the matrix representation of T. (*b*) Find a basis for the kernel of T [null-space]. What is the dimension of this subspace? (*c*) Determine whether $\vec{v} = < -6, -5, -3 > \in Ker(T)$. (*d*) Is T one-to-one? (*e*) Write a spanning set for $Col(A)$, the column space of A. (*f*) What is the dimension of $Col(A)$? (*g*) Is T onto? (*h*) Express, if possible, one of the column vectors of A as a linear combination of the other column vectors of A. Explain in detail. (19 marks: 2, 5, 2 the rest)

(a) Applying $T\langle \hat{e}_j \rangle$ or by direct observation we find

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 1 & 0 & -2 \\ -3 & 3 & 1 \end{pmatrix}$$ 2 marks

(b) We ought to solve $A\vec{x} = \vec{o}$. Then

$$[A|\vec{o}] \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)_{RREF}$$

Note that A is similar to that of Question 1.

$\rightarrow x = 2t,\ y = \frac{5}{3}t,\ z = t \rightarrow \vec{x} = t\langle 2, \frac{5}{3}, 1\rangle$ 3m

$Ker(T) = \{\vec{x} \in \mathbb{R}_3 \mid \vec{x} = t\langle 2, 5/3, 1\rangle\} \rightarrow \mathcal{B}_K = \{\langle 2, \frac{5}{3}, 1\rangle\}$ 1m

1m $\mathcal{D}_K = 1$

(c) Yes, $\vec{v} \in Ker(T)$ 'cos $A\vec{v} = \vec{o}$ 2m

(d) No, T is not injective 'cos $Ker(T) \neq \{\vec{o}\}$ 2m

(e) Simply the cols. A

$\mathcal{Y} = \{\langle -2,1,3\rangle, \langle 3,0,3\rangle, \langle -1,-2,1\rangle\}$ 2m

(f) $\mathcal{D}_{Col} = 2$ 'cos A_{RREF} shows only two l.i. vecs 2m

(g) T is not onto 'cos the col. of A_{ij} don't span $\mathbb{R}_3$. $[\vec{a}_3 = -2\vec{a}_1 - \frac{5}{3}\vec{a}_2]$

(h) From A_{RREF} we see: $\langle -1,-2,1\rangle = -2\langle -2,1,3\rangle - \frac{5}{3}\langle 3,0,3\rangle$ 2m

Carleton University, School of Math & Stats
Math 1104-E [F13] — TEST 2 [40 points]
TUTORIAL SECTION → [ ]

Dr. ℜJ Cova

§1. (12 marks:4 each) A certain linear system has the augmented matrix $\left(\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & h & k \end{array}\right)$. Determine the values of h and k for which the system has (a) no solution, (b) a unique solution, (c) many solutions.

$$\left(\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & h & k \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{array}\right)$$

(a) No solution when $h+6=0 \wedge k-2 \neq 0 \rightarrow \boxed{h=-6,\ k\neq 2}$

(b) Unique solution when $h+6 \neq 0 \wedge k-2 =$ anything $\rightarrow \boxed{h \neq -6,\ \forall k}$

(c) Many solutions when $h+6=0 \wedge k-2=0 \rightarrow \boxed{h=-6,\ k=2}$

§2. (7 marks) Let $\mathcal{B} = \{<-3,1>, <3,2>\}$ be a basis for $\Re_2$. Find the vector $\vec{x} \in \Re_2$ whose coordinate vector relative to $\mathcal{B}$ is $\vec{x}_{\mathcal{B}} = <-1,2>_{\mathcal{B}}$.

Simply, the components of $\vec{x}_{\mathcal{B}}$ are the coefficients of the linear combination

$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$, where $\vec{b}_j$ are the basis vectors

Whence: $\vec{x} = (-1)\langle -3,1\rangle + 2\langle 3,2\rangle \rightarrow \boxed{\vec{x} = \langle 9,3\rangle}$

§3. (*a*) (10 marks) Calculate these determinants: $M = \begin{vmatrix} -1 & 1 & -3 & 2 \\ 3 & 2 & 2 & -6 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & -4 \end{vmatrix}$; $P = \begin{vmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$.

(*b*) (4 marks) May the columns of M serve as a basis for $\Re_4$? And the columns of P? Explain.

(a) $M = -2 \begin{vmatrix} -1 & 1 & -3 & -1 \\ 3 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 2 \end{vmatrix} = 0$, since two columns are equal. 5m

(factor -2 from column # 4)

$P = -\begin{vmatrix} 16 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = +16 \times \frac{1}{4} \times \frac{1}{2} = +2$ 5m

we swapped the second & third rows, then we multiplied the diag. elements because P is diagonal or doubly triangular.

(b) The cols. of M can't be a basis for $\Re_4$ since $|M| = 0$
The cols. of P CAN be a basis for $\Re_4$ since $|P| \neq 0$
2m each

§4. (7 marks) Let S be the parallelogram determined by $< 4, -7 >$ and $< 0, 1 >$. Compute the area of the image of S under the linear transformation with matrix representation $A = \begin{pmatrix} 1 & 1 \\ 7 & 2 \end{pmatrix}$.

Formula: Area $[T(S)] = |A|$ Area (S), where

Area (S) = Absolute value of $\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = 4$ ← 3 m

$|A| = \begin{vmatrix} 1 & 1 \\ 7 & 2 \end{vmatrix} = -5$ ← 1 m

$\Rightarrow$ Area $[T(S)] = 5 \times 4 = 20$ 3 m

Carleton University, School of Math & Stats

Math 1104-E [F13] — TEST 3 [40 points]

TUTORIAL SECTION → [   ]

Dr. ℜJ Cova

§1. (20 marks) Consider the matrix A below. Compute its (a) eigen-values and (b)corresponding eigen-vectors and suitable eigen-bases. (c)Write, if possible, A on its own eigen-basis. If this is not possible, please explain.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 4 \end{pmatrix}. \quad \rightarrow A - \lambda I = \begin{pmatrix} 3-\lambda & 0 & 0 \\ 2 & 1-\lambda & 4 \\ 1 & 0 & 4-\lambda \end{pmatrix}$$

(a) $|A - \lambda I| = 0 \rightarrow (3-\lambda)(1-\lambda)(4-\lambda) = 0 \rightarrow \lambda = 1, 3, 4$

(6 marks, 2 each)

(b) $\lambda = 1 \rightarrow (A - 1I \mid 0) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} u_1 = 0 \\ u_2 = t \\ u_3 = 0 \end{array} \rightarrow \vec{u} = \langle 0, t, 0 \rangle$$

3m

$\mathcal{B}_1 = \{\langle 0, 1, 0 \rangle\}$ 1m

$[t = 1]$

$\lambda = 3 \rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -2 & 4 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} v_1 + v_3 = 0 \\ v_2 = v_3 \\ v_3 = r \end{array} \rightarrow \vec{v} = \langle -r, r, r \rangle$$

$\mathcal{B}_2 = \{\langle -1, 1, 1 \rangle\}$

3+1 m

$[r = 1]$

$\lambda = 4 \rightarrow \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 2 & -3 & 4 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$

3+1 m

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} w_1 = 0 \\ w_2 - \frac{4}{3}w_3 = 0 \\ w_3 = 3p \end{array} \rightarrow \vec{w} = \langle 0, 4p, 3p \rangle$$

$\mathcal{B}_3 = \{\langle 0, 4, 3 \rangle\}$

(c) on its own eigen-basis A becomes diagonal:
(eigenvalues in main diagonal)

$$A_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

2 m

§2. (20 marks) (a)Compute the eigen-values, (b)the corresponding eigen-vectors and suitable eigen-bases for matrix C. (c)Obtain the change of basis matrices $\mathcal{P}$ and $\mathcal{P}^{-1}$. (d)By direct calculation of the similarity transformation $\mathcal{P}^{-1}C\mathcal{P}$ find the diagonal matrix associated with C.

$$C = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \rightarrow C - I\lambda = \begin{pmatrix} -\lambda & 2 \\ -2 & -\lambda \end{pmatrix}$$

(a) $|C - \lambda I| = 0 \Rightarrow \lambda^2 + 4 = 0 \rightarrow \lambda = \pm 2i$ 4m

(b) $\lambda = 2i$: $\left(\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right)$ 3m

$\sim \left(\begin{array}{cc|c} 1 & i & 0 \\ -1 & -i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} v_1 + iv_2 = 0 \\ v_2 = t \end{array} \rightarrow \vec{v} = \langle -it, t \rangle$

$\mathcal{B}_1 = \{\langle -i, 1 \rangle\}$ 1m

$\lambda = 2i$: $\left(\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\rightarrow \begin{array}{l} u_1 - iu_2 = 0 \\ u_2 = r \end{array} \rightarrow \vec{u} = \langle ir, r \rangle$ 3m

$\mathcal{B}_2 = \langle i, 1 \rangle$ 1m

1m

(c) $\mathcal{B} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$; $\mathcal{B}^{-1} = \dfrac{1}{-2i} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$ 4m

Recall: $\mathcal{B}^{-1} = \dfrac{\begin{pmatrix} P_{22} & -P_{12} \\ -P_{21} & P_{11} \end{pmatrix}}{|\mathcal{B}|}$

(d) $\mathcal{B}^{-1} C \mathcal{B} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = C_D$ 3p

§3. Optional BONUS Questions (3 marks each): (a)Explain why the subspace formed by the set $\{\vec{v}\}$ has dimension zero (no basis) if $\vec{v} = \vec{0}$ and dimension one if $\vec{v} \neq \vec{0}$. (b)Explain why $\det(A - \lambda I) = 0$ guarantees that the eigen-equation has no trivial solution.

(Add bonus marks to standard marks. If total is greater than 40, add the excess points to the student's worst test mark)

(a) $\{\vec{0}\}$ has no basis 'cos $\{\vec{0}\}$ is linearly dependent & thus can't serve as a basis. Therefore $\{\vec{0}\}$ has zero dim. On the other hand, $\{\vec{v} \neq \vec{0}\}$ is linearly independent & thus may serve as a basis of dim = 1 ('cos it's just one basis vector). 3m

(b) We know that a homogeneous linear system $M\vec{x} = \vec{0}$ has only the trivial sol. if $|M| \neq 0$. Hence, the condition $|A - \lambda I| = 0$ for the eigen-eq. $(A - \lambda I)\vec{x} = \vec{0}$ eliminates the trivial sol. $\vec{x} = \vec{0}$ from the picture.

3m